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Working Paper - 22/08

Optimal Monetary Policy with Partially Rational Agents

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- July 2008 -

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Abstract

We explore the dynamic behavior of a New Keynesian monetary policy problem with expectations formed, partially, under adaptive learning. We consider two alternative cases: on the first setting, the private economy has the ability to predict rationally real economic conditions (the output gap) but it needs to learn about the future values of the nominal variable (the inflation rate); on the second setup, private agents are fully aware of future inflation rates, however they lack the ability to predict instantly the correct values of the output gap (learning is attached to this variable).

In both cases, we find a simple condition indicating the required learning quality that is needed to guarantee local stability. To achieve convergence to the steady state, the economy does not need to attain full learning efficiency; it just has to secure a minimum learning quality in order to attain the desired long run result.

1 Introduction

To study the impact of monetary policy decisions over the short-run behavior of the economic system, economists typically use a benchmark model known as the New Keynesian monetary policy model [see, e.g., Clarida, Gali and Gertler (1999), Woodford (2003)]. In this model, the central bank solves an optimal control problem (that, here, we consider fully deterministic and defined in discrete time), in which it controls the time path of the nominal interest rate, with the goal of attaining the desired time trajectories for inflation and for the output gap.

A main feature of this monetary policy model is that it attributes a relevant role to present expectations about future values of the endogenous variables. In a deterministic setting, typically a straifghtforward perfect foresight assumption is taken, that is, it is assumed that private agents in the economy possess information and are able to process it optimally such that they do not incur in forecasting mistakes: the expected value at t of the value of a variable at $t+1$ will be the value to observe at $t+1$. This ability of private agents in making forecasts that are not subject to any kind of error is a strong and demanding assumption. As noted by Marcet and Sargent (1989) and Evans and Honkapohja (2001), this assumption would mean a capacity of the agents of the economy in collecting and processing information that is far beyond the capabilities of any human being. Analytically, in the context of monetary policy models, perfect foresight leads to the emergence of a linear system, which is, under conventional assumptions, saddle-path stable.

Here, we add to the perfect foresight hypothesis the possibility of some expectations being formed through an adaptive learning mechanism. More specifically, we consider two alternative cases: on the first setting, the private economy has the ability to predict rationally real economic conditions (the output gap) but it needs to learn about the future values of the nominal variable (the inflation rate); on the second setup, private agents are fully aware of future inflation rates, however they lack the ability to predict instantly the correct values of the output gap (and, thus, they learn them).

The two referred cases are studied in terms of local dynamics (in the vicinity of the unique steady state point that the system contains). In both cases, we find a simple condition indicating the required learning quality that is needed to guarantee stability. To achieve convergence to the steady state (i.e., stability), the economy does not need to attain full learning efficiency; it just has to secure a minimum learning quality in order to attain the desired long run result (although, for reasonable parameter values, we encounter this threshold value close to the maximum efficiency level, i.e., to the learning ability level able to generate a perfect foresight steady state).

The paper is organized as follows. Section 2 presents the benchmark model. Section 3 introduces the learning mechanism and addresses local dynamics. Section 4 concludes.

2 The Optimal Monetary Policy Model

In a given economy, the central bank controls the nominal interest rate ($i_t \geq 0$) and it has, as objective function,

$$V_0 = -\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left[(\pi_t - \pi^*)^2 + a(x_t - x^*)^2\right] \tag{1}$$

The objective function (1) indicates that the monetary authority goal is two-fold: it intends to minimize, assuming an infinite horizon, the difference between the observed inflation rate ($\pi_t \in \mathbb{R}$) and the target that is chosen for this variable (π^*) and the difference between the observed output gap ($x_t \in \mathbb{R}$) and the corresponding target (x^*). The output gap is defined as the difference in logs between effective output and some measure of potential output and the inflation rate is simply the percentage change of the price level. Constant $\beta \in (0, 1)$ is the intertemporal discount factor and parameter $a > 0$ represents the weight of the output gap objective, relatively to the inflation goal, in the monetary authority objective function.

The central bank will maximize (1), subject to two state constraints:

1) IS (investment-savings) equation:

$$x_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t x_{t+1}, \;\; x_0 \text{ given} \tag{2}$$

In equation (2), $\varphi > 0$ measures the sensitivity of the output gap to changes in the real interest rate ($i_t - E_t\pi_{t+1}$) and the operator E_t relates to today's expectations of future values.

2) New Keynesian Phillips curve:

$$\pi_t = \lambda x_t + \beta E_t\pi_{t+1}, \;\; \pi_0 \text{ given} \tag{3}$$

In equation (3), parameter $\lambda \in (0, 1)$ is a measure of price stickiness. The closer this value is to zero, the stronger is the degree of price stickiness or sluggishness.

The maximization of V_0 subject to the two state constraints allows to find an optimal relation between the expected value of the output gap and values at t for both endogenous variables. To arrive to this equation, we build a current-value Hamiltonian function (p_t^x and p_t^π are the shadow prices of the output gap and inflation, respectively),

$$\begin{aligned} H(x_t, \pi_t, i_t, p_t^x, p_t^\pi) = -\frac{1}{2}\left[(\pi_t - \pi^*)^2 + a(x_t - x^*)^2\right] \\ +\beta p_{t+1}^x\varphi\left(i_t - \frac{1}{\beta}\pi_t + \frac{\lambda}{\beta}x_t\right) + \beta p_{t+1}^\pi\left(\frac{1-\beta}{\beta}\pi_t - \frac{\lambda}{\beta}x_t\right) \end{aligned} \tag{4}$$

First-order optimality conditions are,

$$H_i = 0 \Rightarrow \varphi \beta p^x_{t+1} = 0 \tag{5}$$

$$\beta p^x_{t+1} - p^x_t = -H_x \Rightarrow \beta p^x_{t+1} - p^x_t = a(x_t - x^*) - \lambda p^x_{t+1} + \lambda p^\pi_{t+1} \tag{6}$$

$$\beta p^\pi_{t+1} - p^\pi_t = -H_\pi \Rightarrow \beta p^\pi_{t+1} - p^\pi_t = \pi_t - \pi^* + p^x_{t+1} - (1-\beta) p^\pi_{t+1} \tag{7}$$

$$\lim_{t \to +\infty} x_t \beta^t p^x_t = \lim_{t \to +\infty} \pi_t \beta^t p^\pi_t = 0 \text{ (transversality condition)} \tag{8}$$

From the optimality conditions, it is straightforward to obtain the dynamic relation

$$E_t x_{t+1} = \left(1 + \frac{\lambda^2}{a\beta}\right) x_t - \frac{\lambda}{a\beta}\pi_t + \frac{\lambda}{a}\pi^* \tag{9}$$

Equation (9) together with the Phillips curve constitute the two equation - two endogenous variables system that we are interested in addressing analytically. From this system we withdraw two relevant results:[1]

Proposition 1 *The dynamic system obtained from the central bank intertemporal optimization problem has a unique steady state* $(\overline{x}, \overline{\pi}) = \left(\frac{1-\beta}{\lambda}\pi^*; \pi^*\right)$.

Proposition 2 *Under perfect foresight (i.e.,* $E_t x_{t+1} = x_{t+1}$, $E_t \pi_{t+1} = \pi_{t+1}$*), the monetary policy system is saddle-path stable.*

3 Adaptive Learning

For the treatment of the difference equations system, we will consider two alternative settings:

A) Inflation learning / output gap perfect foresight: $E_t x_{t+1} = x_{t+1}$; $E_t \pi_{t+1} = b^\pi_t \pi_t$ (with b^π_t an estimator of inflation based on past information);

B) Output gap learning / inflation perfect foresight: $E_t x_{t+1} = b^x_t x_t$; $E_t \pi_{t+1} = \pi_{t+1}$ (with b^x_t an estimator of the output gap based on past information).

The two settings reveal two interpretations of the reality: in the first, agents can anticipate with precision how real economic conditions will evolve, but they lack the ability to make perfect forecasts regarding inflation. On the second setting, the opposite occurs: nominal or monetary phenomena is predicted with accuracy, but agents need to learn to predict the evolution of the output gap.

[1] Proofs of propositions are omitted in order to save space.

Learning dynamic rules are adapted from Adam, Marcet and Nicolini (2008), and they are as follows:

Case A:

$$b_t^\pi = b_{t-1}^\pi + \sigma_t \left(\frac{\pi_{t-1}}{\pi_{t-2}} - b_{t-1}^\pi \right), \ b_0^\pi \text{ given} \tag{10}$$

Case B:

$$b_t^x = b_{t-1}^x + \sigma_t \left(\frac{x_{t-1}}{x_{t-2}} - b_{t-1}^x \right), \ b_0^x \text{ given} \tag{11}$$

Specially relevant in expressions (10) and (11) is variable $\sigma_t \in [0,1]$. This is known as the gain variable or gain sequence, and it may be interpreted as a measure of the quality of learning. If $\sigma_t \to 0$, then learning is efficient (perfect foresight holds asymptotically). Otherwise, if $\sigma_t \to \overline{\sigma} \in (0,1)$, then some degree of learning inefficiency prevails, meaning that the quality of learning is as better as the lower is $\overline{\sigma}$. We will find that $\overline{\sigma}$ will be decisive in terms of the model's stability properties.

3.1 The Learning Inflation Case (Case A)

Let us begin by case A, in which output gap expectations are formed under perfect foresight and inflation expectations are generated through a process of adaptive learning. Resorting to rule (10), the two-equations system that describes the economy's dynamics is now transformed in a three-equations system, as follows,

$$\begin{cases} x_t = \left(1 + \frac{\lambda^2}{a\beta}\right) x_{t-1} - \frac{\lambda}{a\beta}\pi_{t-1} + \frac{\lambda}{a}\pi^* \\ \pi_t = \frac{x_t}{(1-\sigma_t)\frac{x_{t-1}}{\pi_{t-1}} - \frac{\sigma_t}{\lambda}\left(\beta\frac{\pi_{t-1}}{z_{t-1}} - 1\right)} \\ z_t = \pi_{t-1} \end{cases} \tag{12}$$

Recalling that the system has a unique steady state point, according to proposition 1, we linearize the system in the vicinity of this steady state point, in order to search for local stability conditions under learning,

$$\begin{bmatrix} x_t - \overline{x} \\ \pi_t - \pi^* \\ z_t - \pi^* \end{bmatrix} = \tag{13}$$

$$\begin{bmatrix} 1+\frac{\lambda^2}{a\beta} & -\frac{\lambda}{a\beta} & 0 \\ \frac{\lambda^3}{a\beta(1-\beta)} + \frac{\lambda}{1-\beta}\overline{\sigma} & 1-\overline{\sigma}+\frac{\beta}{1-\beta}\overline{\sigma}-\frac{\lambda^2}{a\beta(1-\beta)} & -\frac{\beta}{1-\beta}\overline{\sigma} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_{t-1}-\overline{x} \\ \pi_{t-1}-\pi^* \\ z_{t-1}-\pi^* \end{bmatrix}$$

Note that the Jacobian matrix in (13) is of dimension three. It is known that for dimension three systems, the following are the necessary stability conditions [see Brooks (2004)],

$$\begin{cases} 1 - Det(J) > 0 \\ 1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0 \\ 1 - Tr(J) + \Sigma M(J) - Det(J) > 0 \\ 1 + Tr(J) + \Sigma M(J) + Det(J) > 0 \end{cases} \tag{14}$$

with $Tr(J)$, $\Sigma M(J)$ and $Det(J)$ representing, respectively, the trace, the sum of principle minors and the determinant of the Jacobian matrix.

These conditions allow to state proposition 3,

Proposition 3 *The optimal monetary policy model with inflation expectations formed under adaptive learning possesses a stable steady state for* $Det(J) < \frac{Tr(J)}{2} - \varkappa + \sqrt{\left(\frac{Tr(J)}{2} - \varkappa\right)^2 + 2 - Tr(J)}$, *with* $\varkappa := \frac{2\lambda^2 + a\beta}{2\lambda^2 + 2a\beta}$*;* $Tr(J) = 2 + \frac{2\beta - 1}{1 - \beta}\overline{\sigma} - \frac{\lambda^2}{a(1-\beta)}$*;* $Det(J) = \frac{\beta}{1-\beta}\left(1 + \frac{\lambda^2}{a\beta}\right)\overline{\sigma}$*.*

The condition in the proposition corresponds solely to the upper bound of the second inequality in (14). This is so because: (i) the lower bound of the second inequality in (14), i.e., $Det(J) > \frac{Tr(J)}{2} - \varkappa - \sqrt{\left(\frac{Tr(J)}{2} - \varkappa\right)^2 + 2 - Tr(J)}$, holds for any positive value of the determinant (the expression in the right side of this inequality is a negative quantity); (ii) verifying that $\frac{Tr(J)}{2} - \varkappa + \sqrt{\left(\frac{Tr(J)}{2} - \varkappa\right)^2 + 2 - Tr(J)} < 1$, we confirm that stability condition $Det(J) < 1$ will hold; (iii) the last two inequalities apply universely, $1 - Tr(J) + \Sigma M(J) - Det(J) > 0 \Leftrightarrow Det(J) > 0$; $1 + Tr(J) + \Sigma M(J) + Det(J) > 0 \Leftrightarrow Det(J) > -\frac{2(\lambda^2 + a\beta)}{2a\beta + 3\lambda^2}Tr(J)$.

A numerical example clarifies the stability result and allows to express it as a constraint on the steady state level of the gain variable. Consider the following benchmark values (quarterly data): $a = 0.25$; $\lambda = 0.024$; $\beta = 0.99$. In this case, $\varkappa := 0.501161$; $Tr(J) = 1.7696 + 98\overline{\sigma}$; $Det(J) = 99.2304\overline{\sigma}$. Applying proposition 3, $\overline{\sigma} < 9.982 \times 10^{-3}$. This is the required stability condition, i.e., the condition that guarantees convergence to $(\overline{x}, \overline{\pi})$.

The stability result may be addressed graphically. The dynamics of the proposed system correspond to the following trace-determinant relation:

$$Det(J) = \frac{\beta}{2\beta - 1}\left(1 + \frac{\lambda^2}{a\beta}\right)\left(\frac{\lambda^2}{a(1-\beta)} - 2\right) + \frac{\beta}{2\beta - 1}\left(1 + \frac{\lambda^2}{a\beta}\right)Tr(J) \tag{15}$$

or, for the considered parameter values, $Det(J) = -1.79182 + 1.012555Tr(J)$.

In figures 1 and 2, we draw this line, along with the various bifurcation lines corresponding to the borders of the stable area. The stable area is the one confined to the intersection of the several conditions. For the case in appreciation, stability will exist from $Det(J) = 0$ (which is equivalent to

$\overline{\sigma} = 0$) until the first bifurcation line is crossed. In figure 1, the general picture does not allow for an immediate perception of which bifurcation line is first crossed, and this is the reason why a detail of figure 1 is presented as figure 2.

**** figures 1,2****

By changing the value of each parameter in 10% (in the case of the discount factor, it is the discount rate that is changed 10%), we can understand how the stability condition is modified. Table 1 reveals that the stability condition is relaxed for a larger a, a lower λ and a lower β.

a	λ	β	Stability condition
0.225	0.024	0.99	$\overline{\sigma} < 9.9791 \times 10^{-3}$
0.275	0.024	0.99	$\overline{\sigma} < 9.9845 \times 10^{-3}$
0.25	0.0216	0.99	$\overline{\sigma} < 9.9873 \times 10^{-3}$
0.25	0.0264	0.99	$\overline{\sigma} < 9.9766 \times 10^{-3}$
0.25	0.024	0.989	$\overline{\sigma} < 1.0982 \times 10^{-2}$
0.25	0.024	0.991	$\overline{\sigma} < 8.9827 \times 10^{-3}$

Table 1 - Sensitivity analysis (case A).

3.2 The Learning Output Gap Case (Case B)

When the private economy is able to forecast perfectly the inflation level for subsequent periods of time, but the output gap is subject to adaptive learning, the system of difference equations that interests us is,

$$\begin{cases} \pi_t = \frac{1}{\beta}\pi_{t-1} - \frac{\lambda}{\beta}x_{t-1} \\ x_t = \frac{\pi^* - \pi_t/\beta}{(1-\sigma_t)\frac{1}{x_{t-1}}\left(\pi^* - \frac{\pi_{t-1}}{\beta}\right) - \frac{a}{\lambda}\frac{x_{t-1}}{v_{t-1}}\sigma_t - \left(\frac{a}{\lambda}+\frac{\lambda}{\beta}\right)\sigma_t} \\ v_t = x_{t-1} \end{cases} \tag{16}$$

Linearizing in the steady state vicinity,

$$\begin{bmatrix} \pi_t - \pi^* \\ x_t - \overline{x} \\ v_t - \overline{x} \end{bmatrix} = \tag{17}$$

$$\begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} & 0 \\ \frac{1}{\lambda\beta} - \frac{1}{\lambda}(1-\overline{\sigma}) & 1-\overline{\sigma}-\frac{1}{\beta}+\frac{a\beta}{\lambda^2}\overline{\sigma} & -\frac{a\beta}{\lambda^2}\overline{\sigma} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi_{t-1}-\pi^* \\ x_{t-1}-\overline{x} \\ v_{t-1}-\overline{x} \end{bmatrix}$$

The main result is now the following,

Proposition 4 *The optimal monetary policy model with output gap expectations formed under adaptive learning possesses a stable steady state for*

$Det(J) < \frac{Tr(J)}{2} - \varrho + \sqrt{\left(\frac{Tr(J)}{2} - \varrho\right)^2 + 2 - Tr(J)},\ with\ \varrho := \frac{1}{2}\left(1 + \frac{\lambda^2}{a}\right);$
$Tr(J) = (1-\overline{\sigma}) + \frac{a\beta}{\lambda^2}\overline{\sigma};\ Det(J) = \frac{a}{\lambda^2}\overline{\sigma}.$

As in the previous case, $Tr(J)$ and $Det(J)$ represent trace and determinant of Jacobian matrix. Once again, we can exclude the first and the last two conditions of (14), since these correspond to universal conditions under the proposed setting ($0 < Det(J) < 1$ and $Det(J) > -\frac{2}{2+\lambda^2/a}Tr(J)$). The lower bound on the second condition is also neglect because it would imply a negative determinant, which is not possible. Thus, the stability condition is again confined to the upper bound of the second stability condition.

For the numerical illustration, we use the same benchmark values as before, which in this case imply $Tr(J) = 1+430.6875\overline{\sigma}$; $Det(J) = 434.0278\overline{\sigma}$. Applying proposition 4, $\overline{\sigma} < 2.298\,7 \times 10^{-3}$. As in case A, stability requires a high learning efficiency, i.e., we must have a steady state gain variable value near zero (near the asymptotic rational expectations equilibrium) in order to guarantee stability.

Figures 3 and 4 present the system's dynamics, once more taking a trace-determinant diagram. The case in appreciation implies $Det(J) = \frac{a}{a\beta-\lambda^2}(Tr(J)-1)$, or, in the specific numeric example, $Det(J) = 1.012\,5Tr(J) - 0.0125$. Figure 3 presents in bold the line referring to the dynamics of the system. The other lines are bifurrcation lines. The system is inside the unit circle as long as $\overline{\sigma}$ remains below a relatively low value; a bifurcation is crossed at $\overline{\sigma} < 2.298\,7 \times 10^{-3}$, a fact that in graphical terms, is better perceived through the detailed presentation of figure 4.

*** *figures 3,4* ***

As in case A, we allow parameter values to vary in order to understand how such changes modify the stability result. Table 2 indicates that the stability condition is relaxed for a lower a and a higher λ, while changes in β are irrelevant.

a	λ	β	Stability condition
0.225	0.024	0.99	$\overline{\sigma} < 2.553\,5 \times 10^{-3}$
0.275	0.024	0.99	$\overline{\sigma} < 2.090\,2 \times 10^{-3}$
0.25	0.0216	0.99	$\overline{\sigma} < 1.862\,8 \times 10^{-3}$
0.25	0.0264	0.99	$\overline{\sigma} < 2.780\,2 \times 10^{-3}$
0.25	0.024	0.981	$\overline{\sigma} < 2.298\,7 \times 10^{-3}$
0.25	0.024	0.991	$\overline{\sigma} < 2.298\,7 \times 10^{-3}$

Table 2 - Sensitivity analysis (case B).

4 Conclusion

We have undertaken a stability analysis of an optimal monetary policy model with inflation (output gap) expectations formed under adaptive learning and output gap (inflation) expectations formed under perfect foresight. In both cases, convergence to the unique steady state requires a minimum learning quality close to the perfect foresight outcome (otherwise, the system is saddle-path stable / unstable). We also found that learning requirements are relaxed with higher a, lower λ and lower β (case A); and with lower a and higher λ (case B). the main policy implication of the model is that public authorities should promote the private economy capabilities regarding the collection and processing of information if they want monetary policy goals to be accomplished.

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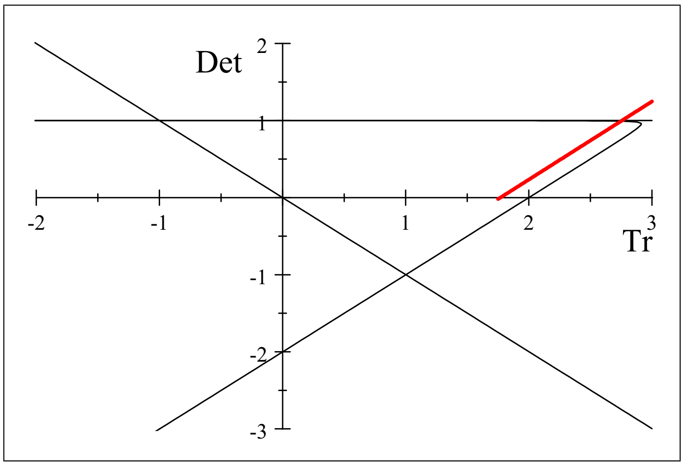
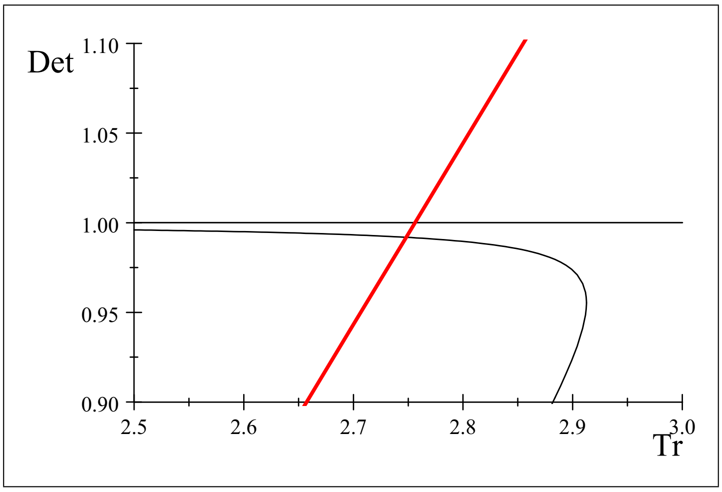
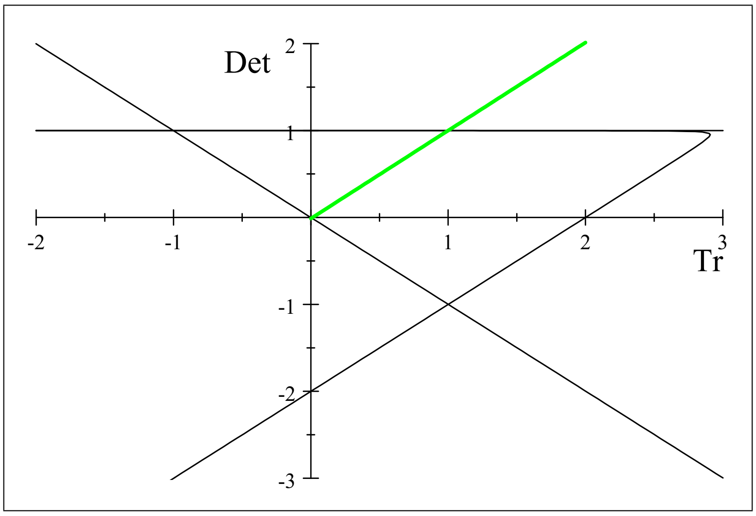
Figures



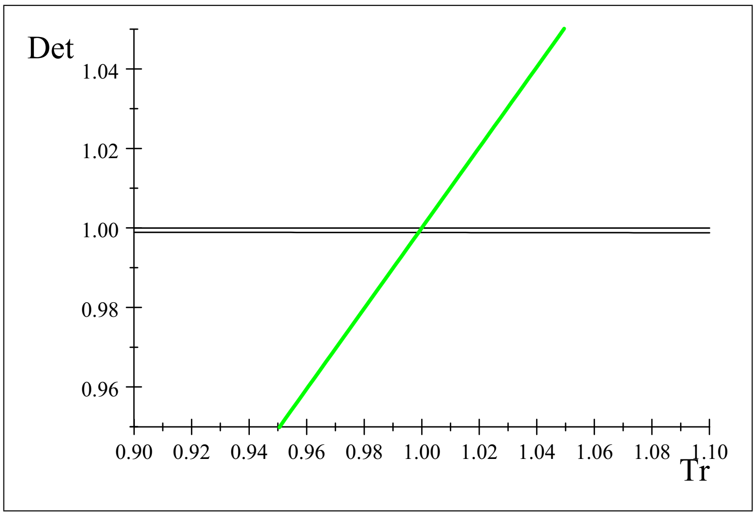
Local dynamics in the trace-determinant diagram (case A)



Local dynamics in the trace-determinant diagram (case A - detail)



Trace-determinant relation (case B)



Trace-determinant relation (case B - detail)