The Dynamics of Learning in Optimal Monetary Policy

Orlando Gomes

Vivaldo M. Mendes

Diana A. Mendes

The Dynamics of Learning in Optimal Monetary Policy

Orlando Gomes [1]   Vivaldo M. Mendes [2]   Diana A. Mendes [3]

— December 2007 —

[1]Corresponding author: Escola Superior de Comunicação Social, Campus de Benfica do IPL, 1549-014 Lisbon, Portugal. Phone number: + 351 93 342 09 15; fax: + 351 217 162 540. e-mail: ogomes@escs.ipl.pt.

[2]ISCTE and UNIDE/ERC, Lisbon. e-mail address: vivaldo.mendes@iscte.pt

[3]IBS Business School and UNIDE/STATMATH – ISCTE, Lisbon. e-mail address: diana.mendes@iscte.pt

Abstract

This paper analyzes the dynamic properties of a standard New Keynesian monetary policy model when private agents expectations are assumed to be formed under a learning mechanism. As pointed out in the literature, learning with decreasing gain estimators tends to lead to convergence to the rational expectations equilibrium; however, under constant gain, persistent learning dynamics prevail and nonlinear trajectories of the state variables may subsist over the long term. By assuming a gain sequence that is asymptotically constant, explicit local and global stability results are presented for two specifications of an optimal monetary policy model. In the first setting, the central bank believes that private agents possess rational expectations; while in the second, the bank incorporates the learning rule in its optimal decisions. In such a framework we find out interesting long term results, in particular, the presence of endogenous business cycles should be stressed as an expected outcome.

Keywords: Learning, Optimal monetary policy, Nonlinear dynamics, Bifurcations and Chaos.

JEL classification: D83, E32, E52, C61.

1 Introduction

A large amount of literature on the formation of macroeconomic expectations through learning has been produced over the last few years. The motivation for this literature can be found in the seminal paper by Marcet and Sargent (1989), who questioned the plausibility of the notion of rational expectations as developed and applied by Muth (1961) and Lucas (1972).

It is well known that under rational expectations economic agents forecast the future without making any systematic errors or, in other words, the observable forecast error corresponds to a white noise process. Such an assumption is extremely strong, in the sense that it implies agents to have all the information that is necessary to make forecasts and, furthermore, that all this information is used optimally. Thus, rational expectations, more than being a plausible assumption about how households and firms predict future values of variables which affect their welfare, are rather a powerful theoretical notion hardly compatible with what agents effectively do in their every day economic activity.

As an answer to the previous criticism, the learning approach began to be conceptualized in the late 1980s. It looks much more reasonable *a priori* to assume that agents may learn about the economic environment over time — such that the process may or may not converge to the rational expectations equilibrium — rather than to impose such equilibrium by simple construction. Thus, instead of knowing the true process underlying the evolution of economic aggregates, the agents will choose a rule that is used to predict future outcomes based on past information. As new information arrives and becomes available, the learning skills improve and the rule is updated.

A continuing process of learning will then probably lead to an asymptotic long run fixed point that may (or may not) differ from the rational expectations equilibrium (REE); if there is convergence, in the long run the individual agents will then have gathered all the information needed to transform the learning rule into an optimal (i.e., rational) rule. This hypothetical convergence to the REE is one of the most relevant properties of learning schemes, as initially pointed out by Marcet and Sargent (1989), or as Beeby, Hall and Henry (2001, p.5) remark, the "attraction of learning then is that it allows agents to make mistakes in the short-run, but not in the long-run".

There are several ways in which learning can be modeled in the field of macroeconomics. The one that has received more attention in the literature is adaptive learning. An extensive survey on macroeconomic issues where adaptive learning is involved is presented in Evans and Honkapohja (2001). This form of learning is the most intuitive one and corresponds essentially to the mechanism mentioned in the previous paragraphs. It is assumed that at each moment agents formulate forecast functions on the basis of all available data. As new data becomes available, these functions are revised over time.

Therefore, full rationality is replaced by a form of bounded rationality. The parameters in the forecast function have to be estimated, and this estimation is generally assumed to follow simple econometric techniques. The process of trial and error that is associated with learning allows, then, to take as perfectly acceptable the argument that rational expectations are just the equilibrium or fixed point of some learning dynamic process. In other words, learning is able to provide an asymptotic justification for the hypothesis of rational expectations.

Results other than the fixed point associated with full rationality are obtainable in adaptive learning settings. Such results may include, as in the two variants presented in this paper, periodic and a-periodic long run cycles. The eventual presence of these cycles is dependent on the different possibilities concerning the choice of a specific learning rule, which involves a gain sequence measuring the sensitivity of estimates to new data. Under the above reasoning, as new information adds to the existing one, the gain should be decreasing, shrinking asymptotically towards zero. Nevertheless, model misspecification or some kind of imperfect knowledge assumption lead us to accept that such gain sequence may not effectively fall to zero. The idea of constant gain learning — i.e., of persistent learning dynamics — is not an unreasonable assumption, and in many settings it can be more appropriate than a simple complete learning scheme with convergence to the REE. After all, it is much easier to find evidence of economic processes where new information is always arriving, thus upgrading permanently the forecast function, than processes capable of being settled off after a given amount of data is gathered for good. Therefore, constant gain forms the crucial element upon which the basic results of our paper are derived.

Most of the learning discussion is related to macroeconomic models where endogenous variables are subject to stochastic disturbances. On such frameworks, two points are usually explored. Firstly, we have the issue of equilibria indeterminacy. As explained in Evans and Honkapohja (2008), indeterminacy implies the existence of a continuum of REE and learning may arise in this context as a way to select the desirable REE.[1] The second point is related to the stability of REE. Errors of forecasting are likely to occur and the mechanisms used to correct such errors may generate an unstable long run outcome. Stability under learning is the central point of discussion in Honkapohja and Mitra (2006), Evans and Honkapohja (2003), Bullard and Mitra (2002), and many others.

In this paper, we discuss a macroeconomic model under learning following some common ground to the existing literature, but we depart from this ground in two crucial ways. As far as the common ground is concerned, the first point in common relates to the benchmark model we use to address ex-

---

[1] On indeterminacy under learning we refer the reader to Carlstrom and Fuerst (2004), Honkapohja and Mitra (2004) and Evans and McGough (2005)

pectations. This is the New Keynesian model with optimal monetary policy, which has been extensively analyzed (under learning), for instance in Bullard and Mitra (2002), Andolfatto, Hendry and Moran (2004), Arifovic, Bullard and Kostyshyna (2007), Preston (2005), Williams (2006), Orphanides and Williams (2003), Evans and Honkapohja (2003, 2006), Gaspar, Smets and Vestin (2006), and Schalling (2003), among others. On this regard, the present study is closer to the work in the three latter references, in the sense that it concentrates on optimal monetary policy, instead of assuming monetary policy under ad-hoc interest rate rules, as in Bullard and Mitra (2002).

The second point of common ground with the existing literature relates to the assumption of a representative private agent that forms expectations concerning (mainly) inflation and also the output gap. Although the assumption of heterogeneous agents is quite appealing when studying stability/determinacy of equilibria — a theme explored in Evans, Honkapohja and Marimon (2001), Giannitsarou (2003), Honkapohja and Mitra (2005, 2006) and Guse (2005) — most of the literature assumes homogeneity of expectations regarding inflation forecasts.

However, our paper departs from the existing literature on expectations under learning by (i) assuming a purely deterministic setup, and (ii) assuming a constant gain in the learning process. The fully deterministic approach has the advantage of securing dynamics that have a unique equilibrium or fixed point. Our main concern is with the long run properties of such equilibrium. One will find complex dynamics in optimal monetary policy models, a result that complements the findings of Grandmont and Laroque (1991), Bullard (1994), Hommes and Sorger (1998), Sorger (1998) and Schonhofer (1999). These papers explore and prove the existence of 'complicated equilibrium trajectories' under least squares learning in a standard version of the overlapping generations growth model. The same is to say that the REE fixed point is replaced by periodic and a-periodic learning equilibria, only possible due to perpetual learning.

A second departure from the existing literature is the type of gain sequence adopted in the paper. Usually, in order to build the forecast functions underlying the learning process, when applying standard econometric techniques (e.g., least squares) the gain sequence is decreasing and falls asymptotically to zero. In this case, convergence to the REE is achieved in standard models. On the contrary, we assume that some notion of incomplete or imperfect knowledge and/or bounded memory may justify a permanent or perpetual process of learning, and therefore long run dynamics would be characterized by a constant gain learning sequence under which some non conventional dynamic results are known to arise. Relevant references on constant gain learning include Orphanides and Williams (2005, 2007), Honkapohja and Mitra (2003), Barucci (1999, 2000) and Timmer-

mann (1995).[2]

The approach in the paper is similar to that of Cellarier (2006) who, under constant gain adaptive learning, studies a neoclassical growth model. In that model, boundedly rational households use learning rules to forecast future prices, physical capital holdings and consumption streams. The analysis focuses on the stability of the steady state. In our model, a similar stability analysis (including the local search for bifurcation points and the global investigation of endogenous fluctuations) is undertaken in a scenario of an optimal monetary policy model under two different settings. Firstly, we consider an extremely simple inflation dynamic difference equation that arises from an environment in which private agents learn but the central bank assumes that agents have perfect foresight. Secondly, we adopt a theoretical structure in which the central bank incorporates on its optimal decision framework the information that private agents effectively learn over time. The quest for bifurcation points in monetary policy models is an issue also discussed in the literature: bifurcations in monetary policy problems solved under non-optimal interest rate rules in a perfect foresight setting are explored in Barnett and He (2002, 2004) and Barnett and Duzhak (2008).

The remainder of the paper is organized as follows. Section 2 briefly presents the benchmark optimal monetary policy model. Section 3 studies the dynamic behavior of the model under learning, assuming that the monetary authority overlooks such learning process by private agents. Section 4 introduces learning from the start, i.e., the central bank incorporates the perception that agents do effectively learn. Finally, section 5 concludes.

2   The optimal monetary policy model

The benchmark model to consider is a fully deterministic version of the New Keynesian monetary policy problem, developed among others in Goodfriend and King (1997), Clarida, Gali and Gertler (1999) and Woodford (2003). The state of the economy is given by two dynamic equations. Aggregate demand is represented by an IS equation, which establishes a relation of opposite sign between the output gap, x_t, and the expected real interest rate, $i_t - E_t\pi_{t+1}$. The output gap is defined as the difference in logs between effective output and some measure of potential output; the inflation rate, π_t, is simply the variation rate of the price level and, in the real interest rate expression, i_t represents the nominal interest rate and E_t is the expectations operator. The complete IS relation is given by the difference equation that follows,

$$x_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t x_{t+1}, \ \ x_0 \text{ given} \tag{1}$$

---

[2]The last reference assumes an endogenous gain sequence, i.e., a gain sequence that is determined by intrinsic economic conditions.

In equation (1), $\varphi > 0$ is an elasticity parameter. As one regards, besides depending on the real interest rate, the contemporaneous value of the output gap also depends on the expected output gap for the subsequent time period.

On the supply side, we assume a New Keynesian Phillips curve, according to which there is a positive relation between the contemporaneous values of inflation and the output gap. The current value of inflation also suffers the influence of the expected value of inflation for the next period. The equation is

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1}, \ \ \pi_0 \text{ given} \tag{2}$$

In equation (2), parameter $\lambda \in (0, 1)$ is a measure of price flexibility. The closer this value is to zero, the stronger is the degree of price stickiness or sluggishness. Constant $\beta \in (0, 1)$ is the intertemporal discount factor.

The monetary authority is supposed to control the value of the nominal interest rate in order to attain some policy goals. We consider that the central bank aims at an inflation rate level π^* and at an output gap x^* (the current practice of monetary authorities points to low but positive inflation and output gap targets). The central bank also attributes different degrees of relevance to the two policy goals. Parameter $a > 0$ will represent the weight of the output gap objective, relatively to the inflation goal, in the monetary authority objective function. Such objective function is the one expressed as follows,

$$V_0 = E_0 \left\{ -\frac{1}{2} \sum_{t=0}^{+\infty} \beta^t \left[(\pi_t - \pi^*)^2 + a(x_t - x^*)^2 \right] \right\} \tag{3}$$

By maximizing V_0 subject to (1) and (2), the central bank chooses the optimal path for the nominal interest rate, that is, the path that allows for minimizing the errors (i.e., the distance between the values of the endogenous state variables and the corresponding targets). The optimization problem yields the following Hamiltonian function (with p_t^x and p_t^π the shadow prices of the output gap and inflation, respectively),

$$H(x_t, \pi_t, i_t, p_t^x, p_t^\pi) = -\frac{1}{2} \left[(\pi_t - \pi^*)^2 + a(x_t - x^*)^2 \right]$$
$$+ \beta p_{t+1}^x \varphi \left(i_t - \frac{1}{\beta} \pi_t + \frac{\lambda}{\beta} x_t \right) + \beta p_{t+1}^\pi \left(\frac{1-\beta}{\beta} \pi_t - \frac{\lambda}{\beta} x_t \right) \tag{4}$$

First-order optimality conditions come,

$$H_i = 0 \Rightarrow \varphi \beta p_{t+1}^x = 0 \tag{5}$$

$$\beta p_{t+1}^x - p_t^x = -H_x \Rightarrow \beta p_{t+1}^x - p_t^x = a(x_t - x^*) - \lambda p_{t+1}^x + \lambda p_{t+1}^\pi \tag{6}$$

$$\beta p_{t+1}^{\pi} - p_t^{\pi} = -H_{\pi} \Rightarrow \beta p_{t+1}^{\pi} - p_t^{\pi} = \pi_t - \pi^* + p_{t+1}^x - (1-\beta)p_{t+1}^{\pi} \quad (7)$$

$$\lim_{t \to +\infty} x_t \beta^t p_t^x = \lim_{t \to +\infty} \pi_t \beta^t p_t^{\pi} = 0 \quad \text{(transversality condition)} \quad (8)$$

Combining the various optimality conditions, one obtains the dynamic relation

$$E_t x_{t+1} = \left(1 + \frac{\lambda^2}{a\beta}\right) x_t - \frac{\lambda}{a\beta}\pi_t + \frac{\lambda}{a}\pi^* \quad (9)$$

The dynamics of the monetary policy problem are addressable with the information given by the Phillips curve in (2) and by equation (9). Two endogenous variables, π_t and x_t, are involved in this system and specific results will depend on how one approaches expectations. We begin by assuming that the central bank believes that private agents have rational expectations and therefore it approaches the system composed by (2) and (9) as if there was perfect foresight in the economy, i.e., $E_t x_{t+1} = x_{t+1}$ and $E_t \pi_{t+1} = \pi_{t+1}$. In this case, solving the monetary policy system implies considering a linear system and, therefore, a perfect coincidence exists between local and global dynamics. In what follows, we will maintain the expectations operators, keeping in mind that the monetary authority solves the model under perfect foresight, but that the private economy forms expectations through learning.

Defining the steady state as the point $(\overline{x}, \overline{\pi})$ such that $\overline{x} \equiv x_t = E_t x_{t+1}$ and $\overline{\pi} \equiv \pi_t = E_t \pi_{t+1}$, one encounters the result $(\overline{x}, \overline{\pi}) = \left(\frac{1-\beta}{\lambda}\pi^*; \pi^*\right)$.

The system can be presented in matricial form,

$$\begin{bmatrix} E_t x_{t+1} - \frac{1-\beta}{\lambda}\pi^* \\ E_t \pi_{t+1} - \pi^* \end{bmatrix} = \begin{bmatrix} 1 + \frac{\lambda^2}{a\beta} & -\frac{\lambda}{a\beta} \\ -\frac{\lambda}{\beta} & \frac{1}{\beta} \end{bmatrix} \cdot \begin{bmatrix} x_t - \frac{1-\beta}{\lambda}\pi^* \\ \pi_t - \pi^* \end{bmatrix} \quad (10)$$

Let J be the Jacobian matrix in system (10). This possesses two eigenvalues, $0 < \varepsilon_1 < 1$ and $\varepsilon_2 > 1$, such that

$$\varepsilon_1, \varepsilon_2 = \frac{a(1+\beta) + \lambda^2}{2a\beta} \mp \sqrt{\left[\frac{a(1+\beta)+\lambda^2}{2a\beta}\right]^2 - \frac{1}{\beta}} \quad (11)$$

Under the assumption that the central bank perceives private expectations about output and inflation as rational ones, the system is characterized by a saddle-path stable equilibrium: in the two-dimensional space that defines the system, there is one stable dimension and one unstable dimension. By computing the eigenvectors associated to each of the eigenvalues, the following expressions are derived:

- Stable trajectory:

$$x_t = \frac{\beta(1 - \varepsilon_1)}{\lambda}\pi^* - \frac{1 - \beta\varepsilon_1}{\lambda}\pi_t \qquad (12)$$

- Unstable trajectory:

$$x_t = -\frac{\beta(\varepsilon_2 - 1)}{\lambda}\pi^* + \frac{\beta\varepsilon_2 - 1}{\lambda}\pi_t \qquad (13)$$

From the analysis of (12) and (13), one observes that the stable trajectory is negatively sloped, independently of parameter values, while the unstable path is positively sloped if condition $\varepsilon_2 > 1/\beta$ is satisfied.

Replacing the output gap expressions in (12) and (13) into the Phillips curve (2), one finds, respectively,

$$E_t\pi_{t+1} = \varepsilon_1\pi_t + (1 - \varepsilon_1)\pi^* \qquad (14)$$

$$E_t\pi_{t+1} = \varepsilon_2\pi_t - (\varepsilon_2 - 1)\pi^* \qquad (15)$$

Equation (14) is stable (it corresponds to the inflation dynamics when the stable path is followed); equation (15) is unstable (it corresponds to the inflation dynamics when the unstable path is followed). Therefore, by choosing an interest rate derived optimally from the assumed maximization problem, the monetary authority guarantees that the steady state value of inflation is the target value that the authority has selected. Nevertheless, only one of the two possible trajectories is stable; this implies that to guarantee a convergence to the steady state inflation rate target, the central bank has not only to choose an optimal interest rate path; it also has to select an initial value of the nominal interest rate that puts the system into the stable arm.

3 Uninformed Central Bank

The central bank intertemporal optimization problem may be solved by assuming that agents have rational expectations, as in the previous section. However, although the central bank may have this belief, private agents might act differently and use some kind of learning rule to form expectations concerning inflation. Here, we follow the mechanism of expectations formation used in Adam, Marcet and Nicolini (2006).

Expectations concerning next period inflation are formed using present and past information. We specify expectations under learning as $E_t\pi_{t+1} = b_t\pi_t$, where b_t is an estimator of inflation based on past information. The mechanism of learning obeys to the rule

$$b_t = b_{t-1} + \sigma_t \left(\frac{\pi_{t-1}}{\pi_{t-2}} - b_{t-1} \right), \quad b_0 \text{ given} \tag{16}$$

Variable σ_t is attached to the notion of gain sequence. We now briefly explain how the gain sequence is approached, and then turn to the dynamic properties of the monetary policy problem under learning.

3.1 The gain sequence

Variable $\sigma_t \in [0, 1]$ is defined as a gain variable or gain sequence. The most commonly used gain sequence is a decreasing sequence such that $\sigma_{t+1} = \sigma_t/(1 + \sigma_t)$, σ_0 given. Under this dynamic relation, as the representative agent collects information, the value of σ_t falls asymptotically towards zero in such a way that rational expectations / perfect foresight holds in the long run. The idea of gain is better understood by defining variable $\alpha_t \equiv 1/\sigma_t$, $\alpha_t \geq 1$. For this, $\alpha_{t+1} = \alpha_t + 1$, i.e., the representative agent improves her prediction at each time moment, endlessly, as new information arrives and the individual effectively learns.

Here, we modify the gain sequence in order to allow for long term constant gain learning (above zero). We assume that agents have finite memory and therefore they learn in each period but they also lose some of the already stored information. Therefore, we modify the above equation in order to include a loss term in the gain expression. Considering that the loss is subject to decreasing marginal returns, we take the expression $\alpha_{t+1} = \alpha_t + 1 - \delta \ln \alpha_t$, with $\delta > 0$ a parameter that measures the extent of memory loss. The equation regarding variable σ_t now comes

$$\sigma_{t+1} = \frac{\sigma_t}{1 + \sigma_t + \delta \sigma_t \ln \sigma_t} \tag{17}$$

The implications of the inclusion of the loss term are significant. First, for the decreasing gain case $\delta = 0$, equation (17) has a unique steady state point, that is stable: $\bar{\sigma}^0 = 0$. For $\delta > 0$, the steady state $\bar{\sigma}^0 = 0$ continues to exist but is no longer stable. Another steady state point arises, $0 < \bar{\sigma} < 1$, which is stable: the gain sequence converges to this value in the long run. Such value corresponds to the nontrivial solution of $\bar{\sigma} \equiv \sigma_t = \sigma_{t+1}$, which is $\bar{\sigma} = \exp(-1/\delta)$.

The value of parameter δ must be bounded from above given that the condition $\sigma_{t+1} \leq 1$ has to be satisfied. Noticing that the maximum of σ_{t+1} is obtained when $\sigma_t = 1/\delta$, then it is straightforward to impose the boundary $\delta \leq \exp(1)$. Figures 1 and 2 present phase diagrams for both cases.[3] Figure 1 respects to the case in which no loss term is introduced in the gain sequence,

---

[3] All the figures presented in this paper, with exception of figure 3, are drawn using IDMC software (interactive Dynamical Model Calculator). This is a free software program available at www.dss.uniud.it/nonlinear, and copyright of Marji Lines and Alfredo Medio.

while figure 2 takes in consideration such loss element; to illustrate this case, we take $\delta = 1$ (a value that implies $\overline{\sigma} = 0.3679$).

*** figures 1, 2 ***

Note the similarities and the differences between figures 1 and 2. For $\sigma_t = 0$ and $\sigma_t = 1$, σ_{t+1} assumes the same values in both cases. The introduction of memory loss in the learning process increases the concavity of the temporal relation of σ_t, making it possible to arrive to a second, stable, steady state. This second steady state will be located as much to the right (higher values of $\overline{\sigma}$) as the larger is the value of δ.

In practice, the above arguments furnish a rationale for the consideration of long term constant gain learning.

3.2 Local stability

Let us recover the optimal monetary policy problem. Under the specified setting, the monetary authority sets the interest rate in an optimal trajectory, which has two arms, one stable, (14), and the other unstable, (15). Although these equations arise as the fruit of a central bank assumption of perfect foresight, the private economy effectively learns, and therefore the estimator b_t may be presented as $b_t = \frac{E_t \pi_{t+1}}{\pi_t} = \varepsilon_1 + (1 - \varepsilon_1)\frac{\pi^*}{\pi_t}$ (if the stable trajectory is followed) and $b_t = \frac{E_t \pi_{t+1}}{\pi_t} = \varepsilon_2 - (\varepsilon_2 - 1)\frac{\pi^*}{\pi_t}$ (if the unstable trajectory is followed). Replacing these expressions in (16), one arrives to the following systems of equations,

$$\begin{cases} \pi_{t+1} = \frac{(1-\varepsilon_i)\pi^*}{\sigma_{t+1}\left(\frac{\pi_t}{z_t} - \varepsilon_i\right) + (1-\sigma_{t+1})(1-\varepsilon_i)\frac{\pi^*}{\pi_t}} & i = 1, 2 \\ z_{t+1} = \pi_t \end{cases} \quad (18)$$

Variable z_t is defined as the inflation rate in period $t - 1$. To synthesize, we must stress that system (18) characterizes the admissible inflation rate paths when (i) the central bank adopts an optimal interest rate rule, assuming that private agents are fully rational regarding their expectations; (ii) agents predict inflation rates under a learning scheme.

Each system has a unique steady state point $\overline{\pi} = \pi^*$; this is the same steady state for inflation one finds in the exclusively perfect foresight case. In the vicinity of this steady state we can study the stability of the system. Linearization in the steady state vicinity leads to the matricial presentation

$$\begin{bmatrix} \pi_{t+1} - \pi^* \\ z_{t+1} - \pi^* \end{bmatrix} = \begin{bmatrix} (1 - \overline{\sigma}) - \frac{\overline{\sigma}}{1-\varepsilon_i} & \frac{\overline{\sigma}}{1-\varepsilon_i} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \pi_t - \pi^* \\ z_t - \pi^* \end{bmatrix}, \ i = 1, 2 \quad (19)$$

A first relevant result is straightforward to obtain from (19),

Proposition 1 *In the optimal monetary policy problem in which the monetary authority overlooks the evidence that the private economy forms expectations through learning, the following local stability results are obtained:*

Case 1. Under (14),

- *If $\overline{\sigma} < \frac{2(1-\varepsilon_1)}{3-\varepsilon_1}$, the system is stable;*

- *If $\overline{\sigma} = \frac{2(1-\varepsilon_1)}{3-\varepsilon_1}$, the system undergoes a flip bifurcation;*

- *If $\overline{\sigma} > \frac{2(1-\varepsilon_1)}{3-\varepsilon_1}$, the system is saddle-path stable.*

Case 2. Under (15),

- *If $\overline{\sigma} < \varepsilon_2 - 1$, the system is stable;*

- *If $\overline{\sigma} = \varepsilon_2 - 1$, the system undergoes a Neimark-Sacker bifurcation;*

- *If $\overline{\sigma} > \varepsilon_2 - 1$, the system is unstable.*

Proof. Trace and determinant of the Jacobian matrix in system (19) are $Tr(J) = (1-\overline{\sigma}) - \frac{\overline{\sigma}}{1-\varepsilon_i}$ and $Det(J) = -\frac{\overline{\sigma}}{1-\varepsilon_i}$. Stability conditions of two-dimensional discrete time systems are the following: $1 - Tr(J) + Det(J) > 0$, $1 + Tr(J) + Det(J) > 0$ and $1 - Det(J) > 0$. These expressions correspond, in the present case, respectively to $\overline{\sigma} > 0$, $2 - \overline{\sigma} - 2\frac{\overline{\sigma}}{1-\varepsilon_i} > 0$ and $1 + \frac{\overline{\sigma}}{1-\varepsilon_i} > 0$. For $i = 1$, the first and the third inequalities are satisfied; the second requires $\overline{\sigma} < \frac{2(1-\varepsilon_1)}{3-\varepsilon_1}$, as specified in the proposition. If the opposite condition holds, then the system is saddle-path stable [because condition $1 + Tr(J) + Det(J) > 0$ is violated]. In the point in which $1 + Tr(J) + Det(J) = 0$, the system undergoes a flip bifurcation. [4]

For $i = 2$, the first and the second stability conditions are satisfied, while the third requires $\overline{\sigma} < \varepsilon_2 - 1$. If $\overline{\sigma} > \varepsilon_2 - 1$, then $Det(J) > 1$, and therefore the system falls in the instability region. When $\overline{\sigma} = \varepsilon_2 - 1$ (i.e., $Det(J) = 1$), the eigenvalues of the Jacobian matrix turn into two complex conjugate values with modulus equal to 1, and the system undergoes a Neimark-Sacker bifurcation ∎

The result in proposition 1 is presentable graphically. If we combine the trace and determinant expressions of the Jacobian matrix in (19), the equation $Det(J) = Tr(J) - (1 - \overline{\sigma})$ is obtained. This relation is depicted graphically in figure 3.

*** figure 3 ***

---

[4] See Medio and Lines (2001), chapter 5, for a detailed discussion of the conditions characterizing the presence of local bifurcations.

Figure 3 represents the simple inflation dynamics learning framework in the trace-determinant diagram. The three lines that form the inverted triangle are bifurcation lines. The area inside the triangle corresponds to the region of stability (two eigenvalues inside the unit circle). The bold line relates to the location of system (19) in terms of the trace-determinant relation. Note that we can separate this line into two segments: for $Det(J) < 0$, we are over the stable arm (14), given that the determinant of the Jacobian matrix is negative as long as we consider a negative $1 - \varepsilon_1$ value. For $Det(J) < 0$, trajectory (15) is taken, since the determinant is positive for $|\varepsilon_2| > 1$. The dynamic results are the consequence of a trace-determinant equation that is parallel to $1 - Tr(J) + Det(J) = 0$, and that locates to the left of this line in the represented diagram.

Especially relevant is the fact that, in any of the cases in proposition 1, stability holds for low values of $\bar{\sigma}$ (near zero). This means that the learning process does not need to be fully efficient (i.e., to converge to the REE) to lead to the stable outcome of rational expectations. Some memory loss is admissible, without this implying a departure from the benchmark result (in the case, this is a convergence to the inflation rate target set by the central bank). When learning inefficiency passes a given threshold (the ones referred in the proposition), then inflation stability is lost, and inflation does not converge any longer to the specified target. This result is economically relevant: it says that agents do not need to be completely efficient when learning, but they need to be almost efficient in order to be possible to attain the desired policy result.

3.3 Global dynamics

Local dynamics indicate that we are in the presence of points of bifurcation. When considering any of the equations (14) and (15), the introduction of the learning mechanism induces the presence of a change in the qualitative nature of the dynamics as one varies the long run value of the gain variable. Recall that such value is directly related to the strength of memory loss in the gain sequence (i.e., with the value of parameter δ). Given the nonlinear nature of the first equation in system (18), one might expect such shifts in the topological properties of the model to produce endogenous fluctuations. In this sub-section, we take some reasonable parameter values to explore the global properties of the system. It is found that as one passes from a local area of stability to an area of saddle-path stability or instability, this is translated, in terms of global dynamics, as the transition from a fixed-point steady state into areas of periodic and a-periodic cycles that exist in a given area before instability sets in.

To address global dynamics, we take the loss parameter δ as the bifurcation parameter, because the former represents the major innovation in the learning process considered in this paper. Remember that $\bar{\sigma} = \exp(-1/\delta)$,

and therefore there is a one to one correspondence between δ and $\bar{\sigma}$. Relatively to the other parameters of the model, we consider values that make sense for a quarterly analysis. First, the inflation rate target is set at $\pi^* = 0.005$ (prices should rise 0.5% a quarter). Second, we consider that the concern of the central bank with the output gap goal is 25% of the concern with inflation ($a = 0.25$). Third, the values for the price stickiness parameter and the discount factor are taken from Rotemberg and Woodford (1997) and Woodford (2003), chapter 5: $\lambda = 0.024$ and $\beta = 0.99$ (this discount factor corresponds to a gross discount rate of 1.01%). Recall that the eigenvalues ε_1 and ε_2 are the ones in (11); hence, for the assumed parameter specifications, one has $\varepsilon_1 = 0.9576$ and $\varepsilon_2 = 1.0549$. In the graphical presentation that follows we assume the initial values $\pi_0 = z_0 = 0.01$ (the initial value of σ_t is irrelevant as long as $\sigma_t \in (0,1)$).

The graphical analysis undertaken here involves presenting bifurcation diagrams, associating the inflation rate with the parameter δ, for both values of ε_i (figures 4 and 5). We observe that for low values of the memory loss (i.e., for low values of the long term gain variable), a stable fixed point is obtained. This confirms that if we are near the REE long term outcome, then the system is stable. As we depart from such outcome, a two-period cycle becomes dominant and regions of a-periodic motion will also arise. These, however, are relatively small. Chaotic motion is formed, in both cases, through a period doubling process.

*** figures 4,5 ***

The diagrams in figures 4 and 5 can be analyzed together with the local dynamics results in proposition 1. For $\varepsilon_1 = 0.9576$, the system is stable if the steady state value of σ_t is lower than 0.0415, which is obtained when $\delta < 0.3143$. A bifurcation occurs at $\delta = 0.3143$, and this can be confirmed by observing figure 4. To the right of this point, local dynamics led to a result of saddle-path stability, that we verify to be a region of cyclical motion. In what concerns the second case, $\varepsilon_2 = 1.0549$, the Neimark-Sacker bifurcation occurs when $\bar{\sigma} = 0.0549$, i.e., $\delta = 0.3446$. To the left of this point we have stability, and to the right instability prevails (locally) and cycles are evidenced (globally).

To highlight the presence of chaotic motion in optimal monetary policy under asymptotic constant gain learning, we choose a value for δ that in figure 5 clearly corresponds to a point of a-periodic cyclicity. For this we draw the long term series of inflation (figure 6). The value of the memory loss parameter is $\delta = 0.97$ (which corresponds to $\bar{\sigma} = 0.3567$) and we assume $\varepsilon_2 = 1.0549$.

*** figure 6***

The presence of chaos can be confirmed by computing Lyapunov characteristic exponents (LCEs). LCEs are a measure of sensitive dependence on initial conditions (SDIC) or divergence of nearby orbits. In a two dimensional system as ours, there exist two LCEs; chaotic motion (SDIC) is revealed if one of the characteristic exponents is above zero, what is effectively true: for the case $\varepsilon_1 = 0.9576$ and $\delta = 1.375$, the computed LCEs are $LCE_1 = -0.63$ and $LCE_2 = 0.12$; for $\varepsilon_2 = 1.0549$ and $\delta = 0.97$, the LCEs come $LCE_1 = -0.52$ and $LCE_2 = 0.08$.[5] The positive exponent indicates that two inflation trajectories starting from points close together will follow completely distinct paths, implying the chaotic result that we have characterized.

The policy implication that one takes from the previous graphical illustration is that although the central bank pursues an optimal policy and it aims at price stability, if the private agents form expectations through learning and the monetary authority does not take into account such learning process, then endogenous cycles may be generated meaning that inflation is not stable, although it fluctuates around a stable value. This inflation fluctuation process requires that as agents learn they also lose past information when estimating future values of key variables (in the case, just inflation).

4   Informed Central Bank

In this section, we sophisticate the model by assuming that the monetary authority is fully aware of the learning mechanism adopted by the private economy in order to predict future values of inflation and of the output gap. We also assume that the learning process is similar for each one of these two variables. The only difference concerns the gain sequence in the sense that it does not need to be exactly the same for both processes (more specifically, we may consider different loss parameters δ^x and δ^π).

By solving the optimal problem, we have arrived at the system

$$\begin{cases} E_t x_{t+1} = \left(1 + \frac{\lambda^2}{a\beta}\right) x_t - \frac{\lambda}{a\beta}\pi_t + \frac{\lambda}{a}\pi^* \\ E_t \pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{\lambda}{\beta}x_t \end{cases} \tag{20}$$

Considering learning, we now take $E_t x_{t+1} = b_t^x x_t$ and $E_t \pi_{t+1} = b_t^\pi \pi_t$, with b_t^x and b_t^π the estimators of the output gap and of the inflation rate respectively. Learning rules are,

$$b_t^x = b_{t-1}^x + \mu_t \left(\frac{x_{t-1}}{x_{t-2}} - b_{t-1}^x\right), \quad b_0^x \text{ given} \tag{21}$$

$$b_t^\pi = b_{t-1}^\pi + \sigma_t \left(\frac{\pi_{t-1}}{\pi_{t-2}} - b_{t-1}^\pi\right), \quad b_0^\pi \text{ given} \tag{22}$$

---

[5] The LCEs are computed also by resorting to the iDMC software.

Equation (22) is identical to equation (16), while equation (21) encloses a similar learning process for the output gap. The gain variable μ_t assumes values in the interval $[0,1]$ and it may be subject to a dynamic process similar to the one characterized in the previous section.

Noticing that $b_t^x = \frac{E_t x_{t+1}}{x_t} = \left(1 + \frac{\lambda^2}{a\beta}\right) - \frac{\lambda}{a\beta}\frac{\pi_t}{x_t} + \frac{\lambda}{a}\frac{\pi^*}{x_t}$ and $b_t^\pi = \frac{E_t \pi_{t+1}}{\pi_t} = \frac{1}{\beta} - \frac{\lambda}{\beta}\frac{x_t}{\pi_t}$, we arrive to the following system of equations,

$$\begin{cases} x_{t+1} = \frac{\pi^*}{1/(\beta\Theta_t) - a\Gamma_t/\lambda} \\ \pi_{t+1} = \frac{\pi^*}{1/\beta - a\Theta_t\Gamma_t/\lambda} \\ m_{t+1} = x_t \\ z_{t+1} = \pi_t \end{cases} \tag{23}$$

System (23) is built using the following definitions: $m_t \equiv x_{t-1}$, $z_t \equiv \pi_{t-1}$, $\Theta_t \equiv (1 - \sigma_{t+1})\frac{x_t}{\pi_t} + \frac{1}{\lambda}\sigma_{t+1}\left(1 - \beta\frac{\pi_t}{z_t}\right)$, $\Gamma_t \equiv (1 - \mu_{t+1})\left(\frac{\lambda}{a\beta}\frac{\pi_t}{x_t} - \frac{\lambda}{a}\frac{\pi^*}{x_t}\right) - \mu_{t+1}\left(\frac{x_t}{m_t} - \left(1 + \frac{\lambda^2}{a\beta}\right)\right)$.

Solving $\overline{x} \equiv x_{t+1} = x_t = m_t$ and $\overline{\pi} \equiv \pi_{t+1} = \pi_t = z_t$, one computes a unique steady state output gap/inflation pair, which is identical to the one already obtained for the case without learning and the case of a simple inflation rate equation dynamics: $(\overline{x}, \overline{\pi}) = \left(\frac{1-\beta}{\lambda}\pi^*; \pi^*\right)$.

Local dynamics are addressable through the linearization of system (23) around the steady state. The system is

$$\begin{bmatrix} x_{t+1} - \overline{x} \\ \pi_{t+1} - \pi^* \\ m_{t+1} - \overline{x} \\ z_{t+1} - \pi^* \end{bmatrix} = J \begin{bmatrix} x_t - \overline{x} \\ \pi_t - \pi^* \\ m_t - \overline{x} \\ z_t - \pi^* \end{bmatrix} \tag{24}$$

with $J = $

$$\begin{bmatrix} -\frac{1}{\beta}(1-\overline{\sigma}) - \frac{1-\beta}{\beta}(1-\overline{\mu}) - \frac{(1-\beta)a}{\lambda^2}\overline{\mu} & \frac{1-\beta}{\beta\lambda}(1-\overline{\sigma}) + \frac{1}{\lambda}\overline{\sigma} + \frac{1-\beta}{\beta\lambda}(1-\overline{\mu}) & \frac{(1-\beta)a}{\lambda^2}\overline{\mu} & -\frac{1}{\lambda}\overline{\sigma} \\ -\frac{\lambda}{\beta}(1-\overline{\mu}) - \frac{a}{\lambda}\overline{\mu} + \frac{\lambda}{\beta}(1-\overline{\sigma}) & \frac{1}{\beta}(1-\overline{\mu}) - \frac{1-\beta}{\beta}(1-\overline{\sigma}) - \overline{\sigma} & \frac{a}{\lambda}\overline{\mu} & \overline{\sigma} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Explicit local results are difficult to obtain given the dimension of matrix J and consequently the cumbersome expressions that result from the corresponding eigenvalues. Thus, we focus the stability analysis on a numerical evaluation. Take the benchmark values in the previous section, $a = 0.25$, $\lambda = 0.024$ and $\beta = 0.99$. From a global perspective, one encounters nonlinear results. We present a detail of a bifurcation diagram for inflation with $\overline{\sigma}$ the bifurcation parameter (figure 7). We let $\overline{\mu} = 0.75$ and the values of the

other involved parameters are the ones already used. The selected initial values are $x_0 = m_0 = \pi_0 = z_0 = 0.01$. It is presented a small region in which chaotic motion is evidenced. We also display the time series of inflation, given a combination of parameters generating irregular cycles (figure 8).

*** figures 7, 8***

The presence of chaotic motion is once again addressed through the computation of LCEs. In this case, we have a four dimensional system. As in the two dimensional case, the presence of a unique positive LCE is sufficient to conclude about the presence of divergence of nearby orbits, i.e., the presence of chaotic trajectories. Table 1 presents the Lyapunov exponents for four different long run values of the gain variable σ_t. The values assumed for the various parameters are the same as before, and, as in figures 7 and 8, we assume $\overline{\mu} = 0.75$.

σ	LCEs	Dynamics
0.585	$-0.13, -0, 18, -0.52, -2.47$	Period 6
0.595	$-0.01, -0.16, -0.54, -2.63$	Period 12
0.605	$0.07, -0.13, -0.56, -2.78$	Chaos
0.615	$-0.44, -0.47, -1.34, -1.37$	Period 2

Table 1 - Lyapunov exponents for different values of $\overline{\sigma}$.

Table 1 displays the LCEs of the system for values of $\overline{\sigma}$ close to each other. Although close, these values produce different dynamics (as one observes in figure 7). The table shows a unique positive LCE, in the only of the selected circumstances in which chaotic motion prevails.

5   Conclusion

The paper addresses the dynamics of optimal monetary policy models under adaptive learning. It is known from the literature that this type of learning mechanism in the formation of expectations may lead to nonlinear dynamics and even chaos, but these results have rarely been illustrated in the literature, which has generally focused on indeterminacy and stability issues in problems where stochastic components are included.

In this paper, we have analyzed the nonlinear dynamics in a deterministic version of the New Keynesian model with optimal monetary policy. Two versions were approached. The first one assumed a rather simple inflation dynamic process where private agents learn but the Central bank does not take this into account, while the second considers a full learning process where expectations about inflation and the output gap are modeled under learning with Central Bank awareness. In both cases, bifurcation points were

found, separating regions of stability from saddle-path stability/instability. A global approach to such systems allowed us to perceive that the bifurcations separate regions in terms of their stability, from stable fixed points, to periodic cyclical motion, and ending up in chaotic dynamics. Dynamics with period 2 cycles is a predominant result but regions of higher periodicity cycles and complete a-periodicity are also revealed. Particularly important is that stability is found solely for low values of the gain variable (i.e., near the REE), meaning that a stable fixed point outcome is directly associated with a high quality learning process.

The obtained results seem to corroborate the idea, which is pervasive in the literature, that endogenous cycles in adaptive learning settings can only arise under constant gain. We have also presented a rationale for long term constant gain: a decreasing marginal loss component allows to obtain a stable fixed point value for the gain process that is located somewhere between 0 and 1.

References

[1] Adam, K.; A. Marcet and J. P. Nicolini (2006). "Stock Market Volatility and Learning." *European Central Bank working paper.*

[2] Andolfatto, D.; S. Hendry and K. Moran (2004). "Inflation Expectations and Learning about Monetary Policy." *DNB Staff Reports* 121, Netherlands Central Bank.

[3] Arifovic, J.; J. Bullard and O. Kostychyna (2007). "Social Learning and Monetary Policy Rules." Working paper 2007-007a, *Federal Reserve Bank of St Louis.*

[4] Barnett, W. A. and E. A. Duzhak (2008). "Non-Robust Dynamic Inferences from Macroeconometric Models: Bifurcation Stratification of Confidence Regions." *Physica A*, forthcoming.

[5] Barnett, W. A. and Y. He (2002). "Stabilization Policy as Bifurcation Selection: Would Stabilization Policy Work if the Economy Really Were Unstable?" *Macroeconomic Dynamics*, vol. 6, pp. 713-747.

[6] Barnett, W. A. and Y. He (2004). "Bifurcations in Macroeconomic Models." in S. Dowrick, R. Pitchford and S. Turnovsky (eds), *Economic Growth and Macroeconomic Dynamics: Recent Developments in Economic Theory*, Cambridge, MA: Cambridge University Press, pp. 95-112.

[7] Barucci, E. (1999). "Heterogeneous Beliefs and learning in Forward Looking Economic Models." *Journal of Evolutionary Economics*, vol. 9, pp. 453-464.

[8] Barucci, E. (2000). "Exponentially Fading Memory Learning in Forward-Looking Economic Models." *Journal of Economic Dynamics and Control*, vol. 24, pp. 1027-1046.

[9] Beeby, M.; S. G. Hall and B. Henry (2001). "Rational Expectations and Near Rational Alternatives: How Best to Form Expectations." *European Central Bank working paper series* 086.

[10] Bullard, J. and K. Mitra (2002). "Learning About Monetary Policy Rules." *Journal of Monetary Economics*, vol. 49, pp. 1105-1129.

[11] Bullard, J. B. (1994). "Learning Equilibria." *Journal of Economic Theory*, vol. 64, pp. 468-485.

[12] Carlstrom, C. T. and T. S. Fuerst (2004). "Learning and the Central Bank." *Journal of Monetary Economics*, vol. 51, pp. 327-338.

[13] Cellarier, L. (2006). "Constant Gain Learning and Business Cycles." *Journal of Macroeconomics*, vol. 28, pp. 51-85.

[14] Clarida, R.; J. Gali and M. Gertler (1999). "The Science of Monetary Policy: A New Keynesian Perspective." *Journal of Economic Literature*, vol. 37, pp. 1661-1707.

[15] Evans, G. W. and B. McGough (2005). "Monetary Policy, Indeterminacy and Learning." *Journal of Economic Dynamics and Control*, vol. 29, pp. 1809-1840.

[16] Evans, G. W. and S. Honkapohja (2001). *Learning and Expectations in Macroeconomics*. Princeton University Press: Princeton, New Jersey.

[17] Evans, G. W. and S. Honkapohja (2003). "Adaptive Learning and Monetary Policy Design" *Journal of Money, Credit and Banking*, vol. 35, pp. 1045-1072.

[18] Evans, G. W. and S. Honkapohja (2003). "Expectations and the Stability Problem for Optimal Monetary Policies." *Review of Economic Studies*, vol. 70, pp. 807-824.

[19] Evans, G. W. and S. Honkapohja (2006). "Monetary Policy, Expectations and Commitment." *Scandinavian Journal of Economics*, vol. 108, pp. 15-38.

[20] Evans, G. W. and S. Honkapohja (2008). "Expectations, Learning and Monetary Policy: an Overview of Recent Research." *Centre for Dynamic Macroeconomic Analysis Working Paper Series*, CDMA 08/02.

[21] Evans, G. W.; S. Honkapohja and R. Marimon (2001). "Convergence in Monetary Inflation Models with Heterogeneous Learning Rules." *Macroeconomic Dynamics*, vol. 5, pp. 1-31.

[22] Gaspar, V.; F. Smets and D. Vestin (2006). "Adaptive Learning, Persistence and Optimal Monetary Policy." *Journal of the European Economic Association*, vol. 4, pp. 376-385.

[23] Giannitsarou, C. (2003). "Heterogeneous Learning." *Review of Economic Dynamics*, vol. 6, pp. 885-906.

[24] Goodfriend, M. and R. G. King (1997). "The New Neoclassical Synthesis and the Role of Monetary Policy." in B. Bernanke and J. Rotemberg (eds.), *NBER Macroeconomics Annual* 1997, Cambridge, Mass.: MIT Press, pp. 231-282.

[25] Grandmont, J.M. and G. Laroque (1991). "Economic Dynamics with Learning: Some Instability Examples." in W.A. Barnett, B. Cornet, C. d'Aspermont, J.J. Gabszewicz, and A. Mas Colell (eds.), *Equilibrium Theory and Applications, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*. Cambridge: Cambridge University Press.

[26] Guse, E. (2005). "Learning with Heterogeneous Expectations in an Evolutionary World." *Cambridge Working Papers in Economics*, n° 0547.

[27] Hommes, C. and G. Sorger (1998). "Consistent Expectations Equilibria." *Macroeconomic Dynamics*, vol. 2, pp. 287-321.

[28] Honkapohja, S. and K. Mitra (2003). "Learning with Bounded Memory in Stochastic Models." *Journal of Economic Dynamics and Control*, vol. 27, pp. 1437-1457.

[29] Honkapohja, S. and K. Mitra (2004). "Are Non-Fundamental Equilibria Learnable in Models of Monetary Policy?" *Journal of Monetary Economics*, vol. 51, pp. 1743-1770.

[30] Honkapohja, S. and K. Mitra (2005). "Performance of Monetary policy with Internal Central Bank Forecasting." *Journal of Economic Dynamics and Control*, vol. 29, pp. 627-658.

[31] Honkapohja, S. and K. Mitra (2006). "Learning Stability in Economies with Heterogeneous Agents." *Review of Economic Dynamics*, vol. 9, pp. 284-309.

[32] Lucas, R. E. (1972). "Expectations and the Neutrality of Money." *Journal of Economic Theory*, vol. 4, pp. 103-124.

[33] Marcet, A. and T. Sargent (1989). "Convergence of Least Squares Learning Mechanisms in Self-referential Linear Stochastic Models." *Journal of Economic Theory*, vol. 48, pp. 337-368.

[34] Medio, A. and M. Lines (2001). *Nonlinear Dynamics: a Primer*. Cambridge, UK: Cambridge University Press.

[35] Muth, J. F. (1961). "Rational Expectations and the Theory of Price Movements." *Econometrica*, vol. 29, pp. 315-335.

[36] Orphanides, A. and J. C. Williams (2005). "Inflation Scares and Forecast-Based Monetary Policy." *Review of Economic Dynamics*, vol. 8, pp. 498.527.

[37] Orphanides, A. and J. C. Williams (2005). "The Decline of Activist Stabilization Policy: Natural Rate Misperceptions, Learning and Expectations." *Journal of Economic Dynamics and Control*, vol. 29, pp. 1927-1950.

[38] Orphanides, A. and J. C. Williams (2007). "Robust Monetary policy with Imperfect Knowledge." *Journal of Monetary Economics*, vol. 54, pp. 1406-1435.

[39] Preston, B. (2005). "Learning about Monetary Policy Rules when Long-Run Horizon Expectations Matter." *International Journal of Central Banking*, vol. 1, pp. 81-126.

[40] Rotemberg, J. and M. Woodford (1997). "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy." *NBER Macroeconomics Annual*, vol. 12, pp. 297-346.

[41] Schaling, E. (2003). "Learning, Inflation Expectations and Optimal Monetary Policy." *Bank of Finland research discussion paper* n° 20/2003.

[42] Schonhofer, M. (1999). "Chaotic Learning Equilibria.", *Journal of Economic Theory*, vol. 89, pp. 1-20.

[43] Sorger, G. (1998). "Imperfect Foresight and Chaos: an Example of a Self-Fulfilling Mistake." Journal of Economic Behaviour and Organization, vol. 33, pp. 333-362.

[44] Timmermann, A. (1995). "Volatility Clustering and Mean Reversion of Stock Returns in an Asset Pricing Model with Incomplete Learning." University of California, *Economics working paper* n° 95-63.

[45] Williams, J. C. (2006). "Monetary Policy in a Low Inflation Economy with Learning." *Federal Reserve Bank of San Francisco working paper* 2006-03.

[46] Woodford, M. (2003). *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton, New Jersey: Princeton University Press.
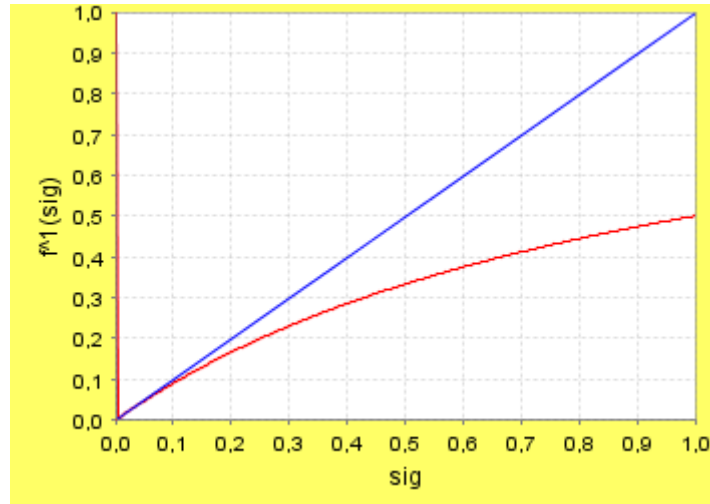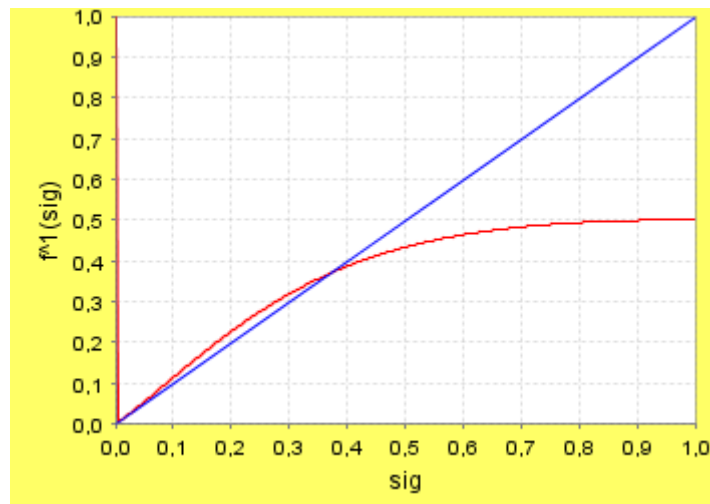
Figures

Figure 1 – Phase diagram for σ_t (δ=0).
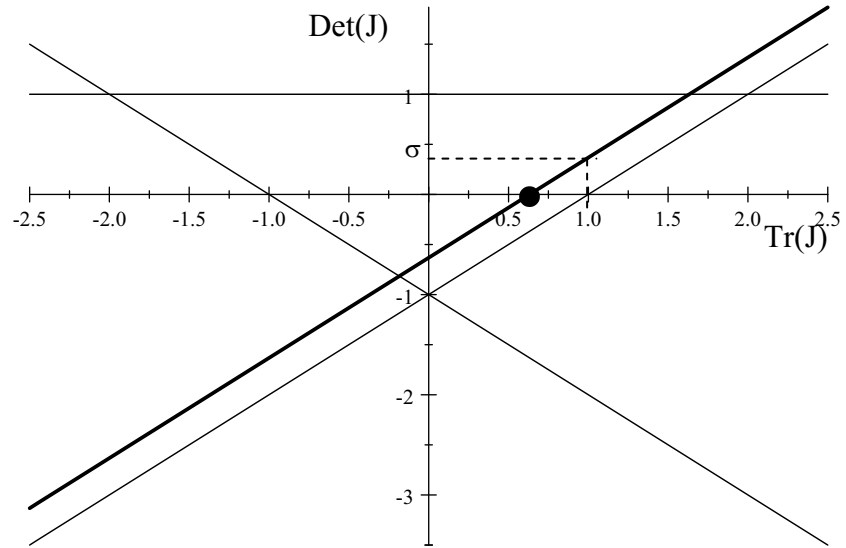

Figure 2 – Phase diagram for σ_t (δ=1).

1

Figure 3 – Local inflation dynamics under learning.
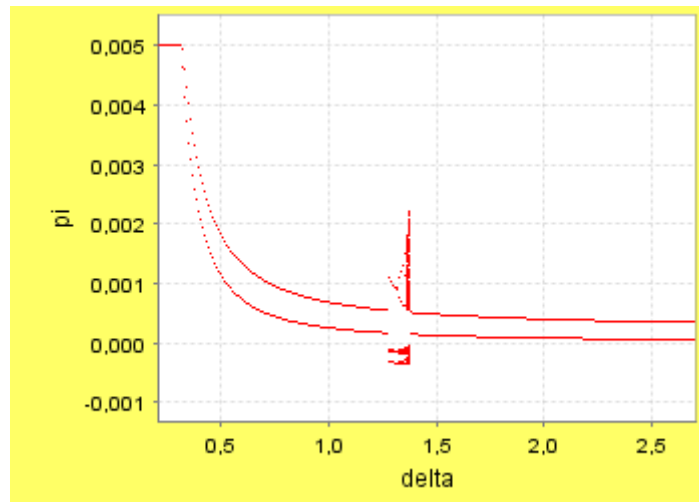
Figure 4 – Bifurcation diagram (π,δ); ε=0.9576.
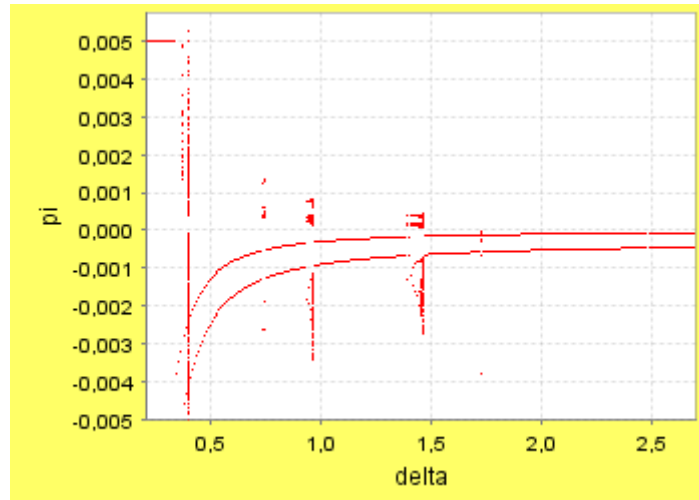
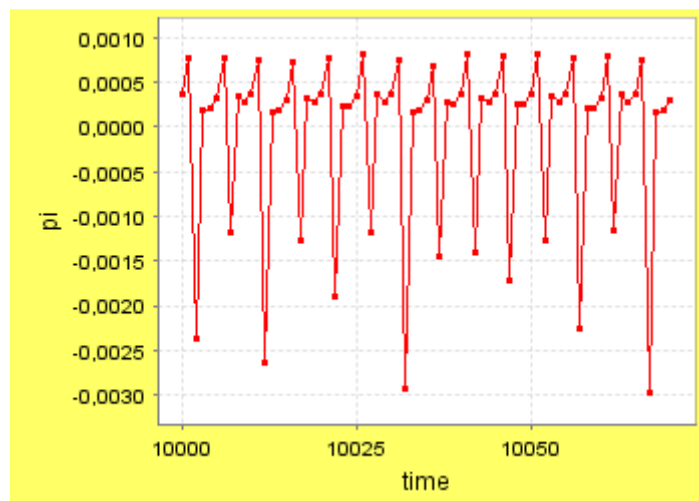**Figure 5 – Bifurcation diagram (π,δ); ε=1.0549.**

Figure 6 – Inflation long run time series; ε=1.0549; δ=0.97.
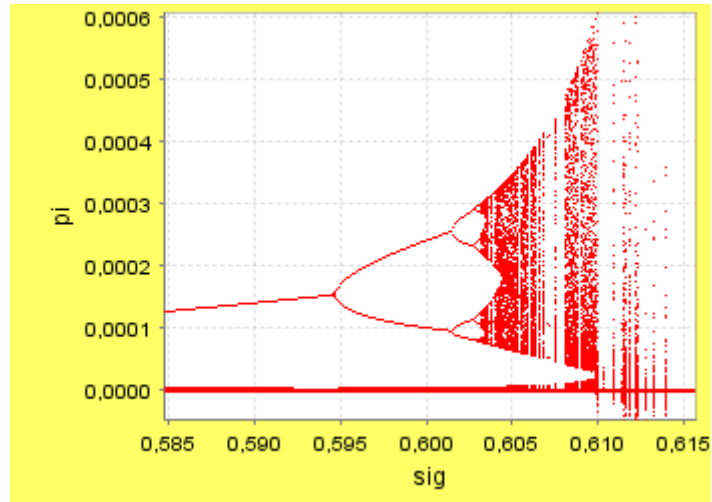
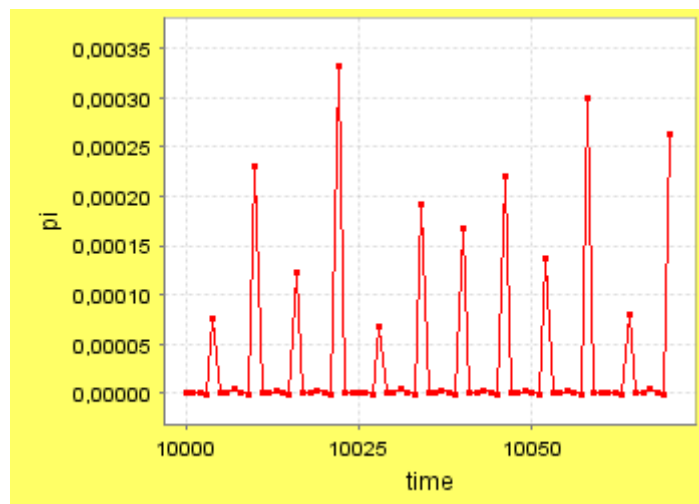**Figure 7 – Detail of the bifurcation diagram in the four dimensional model (π_t,σ); μ=0.75.**

Figure 8 – Long run time series of inflation in the four dimensional model (π_t,x_t). σ=0.605; μ=0.75.