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# Stability under Learning: the Endogenous Growth Problem

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**Abstract**

A local dynamic analysis, in the neighborhood of the steady state, is developed for one and two-sector endogenous growth models. The problem differs from the conventionally assumed growth setups because one considers that expectations concerning the next period value of the control variable (consumption) are formed through adaptive learning. In such scenario, the found stability conditions reveal that convergence to the unique steady state point is feasible if a minimum requirement regarding the quality of learning in the long run equilibrium is fulfilled. Therefore, stability of growth under learning is dependent on the efficiency with which expectations are generated.

**Keywords**: Endogenous growth, Adaptive learning, Stability analysis.

**JEL classification**: O41, C62, D83

# 1 Introduction

When assessing the formation of expectations, a reasonable approach is to consider that some representative agent has to learn about the surrounding environment as time evolves and, therefore, to consider that the agent is unable to produce instantaneous forecasts in a way that is completely accurate. Rational expectations is a too demanding notion, a notion that implicitly considers that the agent has access to all the available information and, furthermore, that this information is processed optimally and immediately. Assuming a setup of bounded rationality, where expectations are formed through a process of trial and error, appears as a more sensible way to incorporate into the analysis the evaluation the agents make about future events, namely when taking into account the explanation of macroeconomic phenomena.

One of the most widely disseminated concepts of learning in the formation of expectations is adaptive learning. Under adaptive learning, at each moment of time the representative agent makes forecasts using the available data and such forecasts are revised over time, as new data becomes available. This form of modelling expectations has been widely used to address macroeconomic phenomena; some of the most outstanding contributions in this field include Evans and Honkapohja (2001), Bullard and Mitra (2002), Basdevant (2005), Preston (2005), Gaspar, Smets and Vestin (2006), Carceles-Poveda and Giannitsarou (2007) and Evans and Honkapohja (2008), just to cite a few influential studies.

A relevant question regarding adaptive learning is whether this learning process is likely to be efficient or not. We may identify a fully efficient learning process as the one in which the forecasting ability converges to a long run rational expectations / perfect foresight outcome.

As pointed out by Honkapohja and Mitra (2003), it is conceivable to assume a complete learning process, i.e., to assume that the economy tends, asymptotically, to settle in a REE (rational expectations equilibrium). Although convergence to the REE is many times pointed out as the rule, namely when one considers agents that are rational and have a strong capacity to learn, this is surely not a universal outcome. The possibility of non convergence may be associated with several misalignments that are relatively easy to identify, and many times not so easy to solve. For instance, Sobel (2000) refers to the efficiency of learning as a basic cost-benefit problem, by stating that optimal learning occurs only as long as the marginal benefit of acquiring and processing information is larger than the corresponding marginal cost.

More specifically, we might identify the asymptotic REE as the natural outcome of learning if three requisites are fulfilled: (i) the model specification is correct (the agent knows what to learn); (ii) the environment is stationary or evolves in a predictable way (i.e., it is unlikely that the agent will be able to converge to a moving target); (iii) the REE is sufficiently rewarding to compensate for the intertemporal costs of learning.

Above all, it is important to realize that agents have a limited knowledge of the environment that surrounds them, and therefore optimal forecasts appear as unrealistic, not only in a transitional dynamics perspective, but also in the long term. After all, if accumulated knowledge and experience increase the accuracy of forecasts, the limited knowledge about the true steady state, a changing environment and the costs / effort involved in learning lead us to think that the REE must be, in many macro frameworks, the exception rather than the rule.

According to the previous reasoning, the most pertinent question is how successful can the agent be if she is not able to generate optimal decisions about future events. In this case, under perpetual learning / imperfect knowledge, the long run outcome will probably differ from the REE. Such deviation may not be substantially significant [as in the monetary policy models of Orphanides and Williams (2005, 2007)], or it can produce completely distinct long term time paths of the endogenous variables, relatively to the rational expectations stable equilibrium benchmark, as in the case of the overlapping generations models analyzed by Bullard (1994), Sorger (1998) or Schonhofer (1999); in these models, the steady state is often characterized by the presence of endogenous business cycles, which emerge when some bifurcation changes the topological properties of the underlying system.

In this paper, our concern will be with the compatibility between an adaptive learning mechanism and the possibility of convergence towards the steady state equilibrium, in the context of endogenous growth models. Two types of growth models are addressed: simple one-sector models of intertemporal utility maximization, in which production is characterized by the presence of constant marginal returns [the simple AK model of Rebelo (1993)], and two-sector growth models, in the line of the ones developed in Lucas (1988), Caballé and Santos (1993), Bond, Wang and Yip (1996) or Gómez (2003, 2004), i.e., models in which physical and human capital are both inputs in the production of final goods and there are constant marginal returns on the education sector.

These growth models are evaluated, as stated, under an adaptive learning framework [we rely, throughout the text, on the adaptive learning rule proposed by Adam, Marcet and Nicolini (2008)]. In particular, the future value of the control variable (consumption) is assumed to be the result of a learning process. The common feature to all the developed models is that stability, at the perfect foresight level, is attainable if some degree of learning quality is ensured, i.e., the representative agent does not have to learn with full efficiency (and, thus, to attain the REE asymptotically), but in turn she faces a minimum learning requirement that allows the stable result to be attained. If such requirement is not fulfilled, the system is unstable and the endogenous variables will diverge from the equilibrium result.

The analysis focuses in the determination of local stability conditions (the analysis of global dynamics leads exactly to the same type of results and, thus, it is neglected), and these highlight the relevance of the variable referring to the measurement of the quality / efficiency of learning. This variable is called the gain sequence and a low value of the steady state level of the gain sequence is needed to guarantee convergence to the pair capital – consumption that subsists in the steady state. A declining gain sequence implies learning efficiency and convergence to a long run zero gain sequence value means that learning was successful in achieving maximum efficiency (and, therefore, the REE).

Given the general equilibrium nature of our models, we should look, as well, to the costs of learning. To improve the formation of expectations, the agents need to apply resources, which are diverted from the production of final goods or knowledge. In this context, a relevant trade-off emerges: to improve expectations (what is crucial to guarantee long run stability) it is necessary to reduce the use of resources in production, implying a lower steady state growth rate. The previous reasoning suggests that the agent should not try to maximize learning efficiency in order to converge to the perfect foresight outcome. Instead, she should be concerned in placing herself in the point where the effort to learn is sufficient to guarantee stability, and place the remaining effort on maximizing utility and achieving the highest possible endogenous growth rate.

The remainder of the paper is organized as follows. Section 2 derives the stability condition for one-sector endogenous growth models (we consider two versions of the model, one with a rival consumption good and the other with a non-rival final good); this section also addresses the costs of learning. Section 3 indicates how one can simplify the treatment of growth models under adaptive learning in order to study rigorously the dynamic properties of the underlying system. Section 4 addresses the two-sector growth model; here, numerical examples have to be used in order to get explicit stability results. Section 5 concludes.

# 2 One-Sector Growth Model

## 2.1 The rival good case

Consider a trivial intertemporal optimization growth problem, where a representative agent maximizes consumption utility subject to a capital accumulation constraint. An infinite horizon is assumed and the future utility is discounted at rate $\rho > 0$. The objective function is $V_0 = \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t U(c_t)$, with $U(c_t) : \mathbb{R}_+ \to \mathbb{R}$ and $c_t$ the level of consumption in moment $t$. The resource constraint is $k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$, $k_0$ given, with $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $k_t$ the stock of physical capital in moment $t$; parameter $\delta > 0$ translates the rate of capital depreciation.

Our benchmark model is an endogenous growth setup and therefore, to allow for sustained growth in the long run, we consider a simple AK production function where marginal returns to capital are constant. Letting $A > 0$ represent the level of technology, the production function is simply $f(k_t) = Ak_t$. Relatively to the utility function, a standard constant intertemporal elasticity of substitution specification is assumed; letting $\theta \in \mathbb{R}_+ \backslash \{1\}$, the function takes the form $U(c_t) = \frac{c_t^{1-\theta}-1}{1-\theta}$. For the particular case $\theta = 1$, we assume $U(c_t) = \ln c_t$. In both cases, the main property a conventional utility function must exhibit, i.e., the presence of decreasing marginal utility, is verified.

In this first setup, final goods are rival: they can either be consumed or, alternatively, returned to production to generate additional value. The current value Hamiltonian function concerning the specified problem is

$$H(k_t, c_t, p_t) = U(c_t) + \left(\frac{1}{1+\rho}\right) E_t p_{t+1}(Ak_t - c_t - \delta k_t)$$

with $p_t$ the shadow-price of capital and $E_t p_{t+1}$ the expected value of the shadow-price of capital for the subsequent period of time. First-order optimality conditions are: (i) $E_t p_{t+1} = (1+\rho)c_t^{-\theta}$; (ii) $(1 + A - \delta)E_t p_{t+1} = (1+\rho)p_t$; (iii) $\lim_{t\to+\infty} k_t \left(\frac{1}{1+\rho}\right)^t p_t = 0$ (transversality condition). The first condition can be presented one period ahead, such that $E_t p_{t+2} = (1 + \rho)\left(E_t c_{t+1}\right)^{-\theta}$; the same for condition (ii): $(1+A-\delta)E_t p_{t+2} = (1+\rho)E_t p_{t+1}$. From these two last expressions, the expected value of consumption for the next period emerges as a constant share of today's level of consumption, i.e., $E_t c_{t+1} = \left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta} c_t$.

Defining $\psi_t := c_t/k_t$ and $E_t\psi_{t+1} := E_t c_{t+1}/k_{t+1}$, the following equation of motion for the consumption-capital ratio is derived,

$$E_t\psi_{t+1} = \frac{\left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta}}{(1+A-\delta) - \psi_t}\psi_t \tag{1}$$

Notice that in the presented problem, the next period value of the state variable (capital) is known with certainty, given the deterministic nature of the resource constraint. In opposition, the next period value of the control variable (consumption) is an expected value, given the established relation between the shadow-price of capital and the level of consumption.

The steady state of the consumption-capital ratio is defined as $\overline{\psi} = \{\overline{\psi} | E_t\psi_{t+1} = \psi_t\}$. Computation reveals a unique value $\overline{\psi} = (1 + A -$

$\delta)\left[1-\frac{(1+A-\delta)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]$; in the case $\theta=1$, the expression simplifies to $\overline{\psi}=\frac{\rho}{1+\rho}(1+A-\delta)$.

Under perfect foresight, i.e., $E_t\psi_{t+1}=\psi_{t+1}$, equation (1) is unstable, given that $\frac{\partial\psi_{t+1}}{\partial\psi_t}=1+\frac{\overline{\psi}}{(1+A-\delta)-\overline{\psi}}>1$. The steady state is accomplished only if the initial value of consumption is located on the stable path (considering variables consumption and capital separately, we would have a two-dimensional system with one stable dimension; this saddle-path stability result implies convergence towards the steady state only if the level of consumption is chosen in order for the stable trajectory to be followed).

Rather than assuming perfect foresight, we consider that expectations are formed through a mechanism of adaptive learning. We consider an estimator $b_t$ such that $E_t\psi_{t+1}=b_t\psi_t$; adaptive learning also requires assuming the dynamic rule $b_t=b_{t-1}+\sigma_t\left(\frac{\psi_{t-1}}{\psi_{t-2}}-b_{t-1}\right)$, $b_0$ given. In this equation, $\sigma_t\in(0,1)$ concerns to the gain sequence, that is, it represents a measure of learning efficiency.

Learning is efficient if $\sigma_t\to 0$ as $t\to+\infty$; in this case, perfect foresight holds asymptotically, i.e., the agent is able to learn how to forecast accurately after a given sequence of time periods. The closer $\sigma_t$ is to 1, in a steady state perspective, the less efficient is the learning process. In this last case, the agent still needs to continue learning in the long run. Of course, a changing environment may imply that the agent needs to continue learning even if she acquires and processes information with efficiency; to simplify the analysis, we will assume that the long run value of the gain sequence is associated with a low learning quality.

We do not explicitly present a dynamic equation for $\sigma_t$; we only specify that there is a unique stable fixed point for $\sigma_t$, $\overline{\sigma}$, that translates the degree of long term learning efficiency. The closer to zero this value is, the more efficient is the learning process.

The value of the estimator is presentable as $b_t=\frac{\left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta}}{(1+A-\delta)-\psi_t}$. Replacing this on the presented learning equation, we arrive to the two-dimensional system,

$$\begin{cases}\psi_t=(1+A-\delta)-\dfrac{\left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta}}{(1-\sigma_t)\frac{\left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta}}{(1+A-\delta)-\psi_{t-1}}+\sigma_t\frac{\psi_{t-1}}{z_{t-1}}}\\ z_t=\psi_{t-1}\end{cases}\qquad(2)$$

The stability properties of system (2) are addressed through a local analysis in the steady state neighborhood. We first consider the simple case $\theta=1$ and, subsequently, we generalize the analysis.

**Proposition 1** *The AK endogenous growth model with a logarithmic utility function is locally stable under adaptive learning if condition $\overline{\sigma}<\rho$ holds.*

**Proof.** The linearization of system (2) in the neighborhood of the steady state yields

$$\begin{bmatrix} \psi_t - \overline{\psi} \\ z_t - \overline{\psi} \end{bmatrix} = \begin{bmatrix} 1 - \overline{\sigma} + \overline{\sigma}/\rho & -\overline{\sigma}/\rho \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_{t-1} - \overline{\psi} \\ z_{t-1} - \overline{\psi} \end{bmatrix}$$

Let $J$ be the Jacobian matrix of the previous linearized system. Stability conditions are:

$1 - Tr(J) + Det(J) = \overline{\sigma} > 0$

$1 + Tr(J) + Det(J) = 2 - \overline{\sigma} + 2\overline{\sigma}/\rho > 0$

$1 - Det(J) = 1 - \overline{\sigma}/\rho > 0$

The first two stability conditions hold for any $\overline{\sigma} \in (0,1)$ and $\rho > 0$. The third condition requires $\overline{\sigma} < \rho$ ■

The result in proposition 1 indicates that a relatively high level of learning efficiency is necessary to guarantee stability (at the found steady state level). The value of $\overline{\sigma}$ must be lower than the intertemporal discount rate and parameters $A$ and $\delta$ do not influence the long term outcome. If $\overline{\sigma} = \rho$, the system undergoes a Neimark-Sacker bifurcation (the eigenvalues of matrix $J$ turn into two complex values with modulus equal to 1) and local instability implies $\overline{\sigma} > \rho$.[1]

Turning to the case $\theta \neq 1$, one finds a more sophisticated stability condition,

**Proposition 2** *The AK endogenous growth model with a generic constant intertemporal elasticity of substitution utility function is locally stable under adaptive learning if condition* $\overline{\sigma} < (1 + A - \delta)^{(\theta-1)/\theta}(1+\rho)^{1/\theta} - 1$ *holds.*

**Proof.** Proceeding as in the proof of proposition 1, we linearize system 2,

$$\begin{bmatrix} \psi_t - \overline{\psi} \\ z_t - \overline{\psi} \end{bmatrix} = \begin{bmatrix} 1 - \overline{\sigma} + \overline{\sigma}/x & -\overline{\sigma}/x \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \psi_{t-1} - \overline{\psi} \\ z_{t-1} - \overline{\psi} \end{bmatrix}$$

with $x := (1 + A - \delta)^{(\theta-1)/\theta}(1+\rho)^{1/\theta} - 1$. Computing stability conditions

$1 - Tr(J) + Det(J) = \overline{\sigma} > 0$

$1 + Tr(J) + Det(J) = 2 - \overline{\sigma} + 2\overline{\sigma}/x > 0$

$1 - Det(J) = 1 - \overline{\sigma}/x > 0$

Because one must have $\overline{\psi} > 0$, then $x > 0$. Thus, once again, the first and the second stability conditions hold, and the third requires $\overline{\sigma} < x$; this is the condition in the proposition ■

[1] Global dynamic results do not differ, in what concerns stability conditions, from the ones derived locally. The stability area is the same and the local instability regions correspond mainly to irregular cycles with no economic meaning [extremely high values (both positive and negative) of the ratio $\psi_t$]. This pattern is found for the various model specifications thorughout the paper; thus, we neglect global dynamics and focus on a local perspective.

The result in proposition 2 nests $\theta = 1$ as a particular case. Note that in the general case the level of technology and the depreciation rate are important parameters for stability. The main result is similar to the previously obtained: a minimum degree of learning efficiency is required; in this case, the set of parameters $(\rho, A, \delta, \theta)$ is relevant to the formation of the threshold value.

The stability outcome of the AK growth model for a rival final good is presentable graphically within a trace-determinant framework. The following relation can be established between trace and determinant of the Jacobian matrix of the linearized version of (2): $Det(J) = Tr(J) - (1 - \overline{\sigma})$; recalling that $Det(J) > 0$, the dynamics of the system is translated graphically by the bold line in figure 1.

*** *figure 1* ***

In figure 1, the area inside the inverted triangle formed by the three bifurcation lines corresponds to the stable area. The system locates in this area if $\overline{\sigma}$ is relatively low. Instability prevails after the Neimark-Sacker bifurcation line ($Det(J) = 1$) is crossed.

## 2.2 The non-rival good case

On a second version of the endogenous growth problem, we assume that the produced final good is non-rival ($\widetilde{k}_t \geq 0$). Thus, the representative agent maximizes $\widetilde{V}_0 = \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t U(\widetilde{k}_t)$ subject to $\widetilde{k}_{t+1} = A\widetilde{k}_t + (1-\delta)\widetilde{k}_t$, $\widetilde{k}_0$ given. The utility function is similarly specified relatively to the first model, i.e., $U(\widetilde{k}_t) = \frac{\widetilde{k}_t^{1-\theta}-1}{1-\theta}$ for $\theta \in \mathbb{R}_+\backslash\{1\}$; the particular case $U(\widetilde{k}_t) = \ln \widetilde{k}_t$ is also addressed.

The optimization problem allows to build the current value Hamiltonian function,

$$H(\widetilde{k}_t, \widetilde{p}_t) = U(\widetilde{k}_t) + \left(\frac{1}{1+\rho}\right) E_t\widetilde{p}_{t+1}(A - \delta)\widetilde{k}_t$$

Once again, $\widetilde{p}_t$ refers to the shadow-price of the final good and $E_t\widetilde{p}_{t+1}$ is the corresponding expected value for the next period.

First-order conditions come: (i) $(1 + A - \delta)E_t\widetilde{p}_{t+1} = (1 + \rho)\left(\widetilde{p}_t - \widetilde{k}_t^{-\theta}\right)$; (ii) $\lim_{t\to+\infty} \widetilde{k}_t \left(\frac{1}{1+\rho}\right)^t \widetilde{p}_t = 0$ (transversality condition). We now define $\varphi_t := \widetilde{p}_t\widetilde{k}_t^{\theta}$ and $E_t\varphi_{t+1} := E_t\widetilde{p}_{t+1}\widetilde{k}_{t+1}^{\theta}$. As a result,

$$E_t\varphi_{t+1} = (1 + A - \delta)^{(\theta-1)}(1 + \rho)(\varphi_t - 1) \tag{3}$$

Letting the steady state value of $\varphi_t$ be $\overline{\varphi} = \{\overline{\varphi}|E_t\varphi_{t+1} = \varphi_t\}$, a unique steady state is found: $\overline{\varphi} = \frac{1+\rho}{(1+\rho)-(1+A-\delta)^{1-\theta}}$. The need to ensure $\overline{\varphi} > 0$ imposes the constraint $(1+\rho) > (1+A-\delta)^{(1-\theta)}$; the condition is automatically verified for $\theta = 1$, case in which $\overline{\varphi} = (1+\rho)/\rho$.

In the case of perfect foresight, the condition implying $\overline{\varphi} > 0$, also signifies that $\partial\varphi_{t+1}/\partial\varphi_t > 1$, i.e., equation (3) exhibits instability. For some $\varphi_0 \neq \overline{\varphi}$, the system does not converge to the steady state. Under adaptive learning, stability properties may be addressed just as in the previous section. First, consider the estimator $\widetilde{b}_t = E_t\varphi_{t+1}/\varphi_t$ and the rule $\widetilde{b}_t = \widetilde{b}_{t-1} + \sigma_t\left(\frac{\varphi_{t-1}}{\varphi_{t-2}} - \widetilde{b}_{t-1}\right)$, $\widetilde{b}_0$ given.

The learning setup yields the following system,

$$\begin{cases} \varphi_t = \dfrac{1}{(1-\sigma_t)/\varphi_{t-1}+\sigma_t\left(1-\frac{(1+A-\delta)^{1-\theta}}{1+\rho}\frac{\varphi_{t-1}}{v_{t-1}}\right)} \\ v_t = \varphi_{t-1} \end{cases} \tag{4}$$

Proposition 3 furnishes the stability result,

**Proposition 3** *The AK endogenous growth model with a non-rival final good is locally stable under condition $\overline{\sigma} < (1+A-\delta)^{(\theta-1)}(1+\rho)^{1/\theta} - 1$. This nests as a special case $\overline{\sigma} < \rho$, for $\theta = 1$.*

**Proof.** Proceeding in the same way as for the propositions in the previous section, we linearize (4) around $\overline{\varphi}$, obtaining the matricial system

$$\begin{bmatrix} \varphi_t - \overline{\varphi} \\ v_t - \overline{\varphi} \end{bmatrix} = \begin{bmatrix} 1-\overline{\sigma}+\overline{\sigma}/\widetilde{x} & -\overline{\sigma}/\widetilde{x} \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varphi_{t-1} - \overline{\varphi} \\ v_{t-1} - \overline{\varphi} \end{bmatrix}$$

with $\widetilde{x} := (1+A-\delta)^{(\theta-1)}(1+\rho)^{1/\theta} - 1$. The system is stable if the following conditions hold for the Jacobian matrix in the system:

$1 - Tr(J) + Det(J) = \overline{\sigma} > 0$

$1 + Tr(J) + Det(J) = 2 - \overline{\sigma} + 2\overline{\sigma}/\widetilde{x} > 0$

$1 - Det(J) = 1 - \overline{\sigma}/\widetilde{x} > 0$

The first two stability conditions are satisfied [note that $\widetilde{x} = (1+A-\delta)^{(\theta-1)}(1+\rho)/\overline{\varphi} > 0$]. The third condition requires $\overline{\sigma} < \widetilde{x}$, as stated in the proposition. If $\theta = 1$, it is straightforward to arrive to the same stability condition as in the rival final good case ■

Qualitatively, results are not radically different from the rival good case. A minimum learning efficiency ceiling is required in order for stability at the perfect foresight level of the endogenous variable to hold. Once again, we obtain the trace-determinant relation $Det(J) = Tr(J) - (1-\overline{\sigma})$ and we observe that $Det(J) > 0$. Figure 1 depicts, also in this circumstance, the dynamic behavior that is implicit in the learning model.

## 2.3 The costs of learning

The pursued stability analysis has suggested that there is a minimum learning efficiency needed to attain convergence to the steady state. If no costs are associated with acquiring and processing information in order to learn, no constraint is placed on how much the representative agent learns, as long as the stability interval is attained. If, to learn, it is necessary to spend resources, then the agent has an advantage in placing herself immediately to the left of the value that guarantees stability, saving resources relatively to a situation of completely efficient learning (i.e., a learning process guaranteeing $\overline{\sigma} = 0$).

Recover our simplest model, the one with a rival good and logarithmic utility. In this case, the steady state growth rate of the economy is $\gamma = (A - \rho - \delta)/(1 + \rho)$, in the absence of learning costs. Now, consider that a certain amount of physical resources $\mu k_t$ is used to learn through time. The resource constraint is changed to $k_{t+1} = Ak_t - c_t - \mu k_t + (1-\delta)k_t$ ($\mu \in (0, 1)$), and the growth rate decreases to $\gamma = (A - \mu - \rho - \delta)/(1 + \rho)$. If the long run value of the gain sequence is dependent on $\mu$, such that, for $\overline{\sigma}(\mu)$, we have $\overline{\sigma}_\mu < 0$, then a trade-off arises: investing in learning improves the learning result (lower $\overline{\sigma}$) but it reduces the long run rate of growth. Furthermore, given that $\overline{\psi} = \frac{\rho}{1+\rho}(1 + A - \mu - \delta)$, diverting resources from the production of goods to learning will imply a smaller long run consumption-capital ratio.

The desirable share of physical resources that should be allocated to learning in our framework is easy to determine: it should be the value that allows for a steady state gain sequence immediately below the discount rate; this is the value that allows for the highest possible steady state growth rate and consumption-capital ratio without compromising stability.

Let us consider a numerical example. We assume the benchmark values of the endogenous growth model taken in Barro and Sala-i-Martin (1995), page 191; all values are per year: $\rho = 0.02$, $\delta = 0.05$, $A = 0.11$ (we also assume that the consumption good is rival and that $\theta = 1$).[2] With these values, the growth rate of the economy in the steady state is $\gamma = 0.0392$ (3.92%) and the consumption-capital ratio is $\overline{\psi} = 0.0208$, i.e., consumption represents 2.08% of the accumulated capital.

Now, assume the following relation between the steady state gain sequence and the share of resources devoted to learning: $\overline{\sigma}(\mu) = s(1 - \mu)^2$, with $0 < s < 1$. Take $s = 0.021$ and consider the following alternatives for $\mu$: $\mu = 0$; $\mu = 0.01$; $\mu = 0.02$; $\mu = 0.03$; and $\mu = 0.04$. Table 1 indicates how learning costs influence the steady state outcome and allows to understand

[2] In Barro and Sala-i-Martin (1995), the value of the technology relates to the human capital sector. Here, for now, we do not have an education sector and, thus, it is the technology of the final goods sector that functions as the engine of growth. Thus, we use that value of technology to characterize the technology conditions of our economy and to obtain a reasonable long term economic growth rate.

where the efficient learning cost is located.

| $\mu$ | $\gamma$ | $\overline{\psi}$ | $\overline{\sigma}$ | Stability ($\overline{\sigma} < 0.02$) |
|---|---|---|---|---|
| 0 | 0.0392 | 0.0208 | 0.021 | $No$ |
| 0.01 | 0.0294 | 0.0206 | 0.0206 | $No$ |
| 0.02 | 0.0196 | 0.0204 | 0.0202 | $No$ |
| 0.03 | 0.0098 | 0.0202 | 0.0198 | $Yes$ |
| 0.04 | 0 | 0.02 | 0.0193 | $Yes$ |

Table 1 - Stability result under different learning investment effort.

Table 1 indicates that the absence of resources diverted from the productive activity to learning benefit the economy both in terms of the steady state growth rate and in terms of the long run level of consumption relatively to the stock of capital. Nevertheless, the absence of investment in learning or a too low investment does not allow to attain the steady state result. Only for $\mu \geq 0.03$ we observe stability. Thus, the representative agent should choose to allocate, in the steady state, 3% of the accumulated resources to improve the learning capabilities. In this way, the steady state is accomplished for the best possible growth rate (0.98%) and the best possible consumption-capital ratio (0.0202). Observe that the steady stat growth rate falls significantly with the increase in the share of capital withdrawn from the goods sector, while the steady state level of the consumption-capital ratio does not suffer significantly with the rise in $\mu$.

# 3 A Two-Step Linearization Procedure

## 3.1 The general procedure

The one sector endogenous growth model analyzed above did not present a too demanding analytical challenge, given the simplicity of the involved expressions. Introducing additional elements into the model (as in the two-sector model of the following section), the simple analysis that we have taken becomes compromised. In this section, we propose a shortcoming that turns possible the analysis. This shortcoming will consist on a two-step linearization procedure. Recall that the analysis is local, in the neighborhood of the steady state, and therefore we may proceed with a linearization process in two phases: first, we linearize the value of the estimator $b_t$ and, then, the obtained system of equations. This procedure will be explained in general terms, and then applied to the one-sector endogenous growth model, to show its viability.

Consider some variable, defined in time, $X_t \in \mathbb{R}$. The expected value of this variable is $E_t X_{t+1} = B_t X_t$, with $B_t$ the adaptive learning estimator: $B_t = B_{t-1} + \sigma_t \left( \frac{X_{t-1}}{X_{t-2}} - B_{t-1} \right)$, $B_0$ given. The gain sequence is defined as previously. Consider that there is a unique steady state point $\overline{X}$. We may linearize function $F(X_t) = E_t X_{t+1}/X_t$ in the steady state neighborhood, to obtain $F(X_t) \simeq 1 + F'(\overline{X})(X_t - \overline{X})$. Replacing the estimator in the adaptive learning rule by function $F$, the following explicit two-dimensional system is obtained:

$$\left\{ \begin{array}{l} X_t = (1-\sigma_t)X_{t-1} + \frac{\sigma_t}{F'(\overline{X})}\left[\frac{X_{t-1}}{Z_{t-1}} - \left(1 - F'(\overline{X})\overline{X}\right)\right] \\ Z_t = X_{t-1} \end{array} \right. \tag{5}$$

The local evaluation of system (5) around $\overline{X}$ leads then to the matricial system

$$\left[ \begin{array}{c} X_t - \overline{X} \\ Z_t - \overline{X} \end{array} \right] = \left[ \begin{array}{cc} 1 - \overline{\sigma} + \overline{\sigma}/(F'(\overline{X})\overline{X}) & -\overline{\sigma}/(F'(\overline{X})\overline{X}) \\ 1 & 0 \end{array} \right] \cdot \left[ \begin{array}{c} X_{t-1} - \overline{X} \\ Z_{t-1} - \overline{X} \end{array} \right] \tag{6}$$

A generic stability condition is presentable,

**Proposition 4** *The value $\overline{X}$ corresponds to a stable steady state value if:*
*(i) $\overline{\sigma} < F'(\overline{X})\overline{X}$ (in the case in which $F'(\overline{X})\overline{X} > 0$);*
*(ii) $\overline{\sigma} < 2F'(\overline{X})\overline{X}/(F'(\overline{X})\overline{X} - 2)$ (in the case in which $F'(\overline{X})\overline{X} < 0$).*

**Proof.** Trace and determinant of the Jacobian matrix of system (6) are, respectively, $Tr(J) = 1 - \overline{\sigma} + \overline{\sigma}/(F'(\overline{X})\overline{X})$ and $Det(J) = \overline{\sigma}/(F'(\overline{X})\overline{X})$. The stability conditions come,
$1 - Tr(J) + Det(J) = \overline{\sigma} > 0$;
$1 + Tr(J) + Det(J) = 2 - \overline{\sigma} + 2\overline{\sigma}/(F'(\overline{X})\overline{X}) > 0$;
$1 - Det(J) = 1 - \overline{\sigma}/(F'(\overline{X})\overline{X}) > 0$.
Assuming $F'(\overline{X})\overline{X} > 0$, the first two stability conditions hold, and the third one requires $\overline{\sigma} < F'(\overline{X})\overline{X}$. Assuming $F'(\overline{X})\overline{X} < 0$, the first and the third stability conditions are verified and the second implies $\overline{\sigma} < 2F'(\overline{X})\overline{X}/(F'(\overline{X})\overline{X} - 2)$ ■

Two points should be emphasized in what respects proposition 4. First, we may complete the dynamic analysis, by inquiring about points of bifurcation. In the case $F'(\overline{X})\overline{X} > 0$, the point $\overline{\sigma} = F'(\overline{X})\overline{X}$ will signify a Neimark-Sacker bifurcation (two complex eigenvalues with real part equal to one) separating stability and instability regions. In the case $F'(\overline{X})\overline{X} < 0$, the point $\overline{\sigma} = 2F'(\overline{X})\overline{X}/(F'(\overline{X})\overline{X} - 2)$ is the frontier between stability and saddle-path stability, and it corresponds to a flip bifurcation (one of the two eigenvalues assumes the value $-1$). If one combines trace and determinant of $J$, we obtain the relation $Det(J) = Tr(J) - (1 - \overline{\sigma})$ (the same relation we had on the endogenous growth problem; the difference relatively to the present setting is that we considered only the case corresponding to $F'(\overline{X})\overline{X} > 0$).

The trace-determinant relation is such that we encounter the dynamics of the model on a parallel line to $1 - Tr(J) + Det(J) = 0$, as in figure 1. If one allows for $F'(\overline{X})\overline{X} < 0$, we may extend that line to negative values of the determinant, and observe that the bifurcation line $1 + Tr(J) + Det(J) = 0$ is crossed in a flip bifurcation point.

The second relevant information that proposition 4 furnishes relates to the intuitive idea that, generically, a stability condition is a condition that indicates which is the maximum value of learning inefficiency, above which the system is no longer stable.

## 3.2 Applying the linearization procedure to the one-sector growth model

In this subsection, we use the previously described procedure to attain the same results one already knows concerning the one-sector endogenous growth model. We begin by looking at the rival good case.

Reconsider the rival good one-sector endogenous growth model analyzed previously. Let us define function $F(\psi_t) := \frac{E_t\psi_{t+1}}{\psi_t} = \frac{\left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta}}{(1+A-\delta)-\psi_t}$. Adopting the local linearization procedure, this function is presentable as $F(\psi_t) \simeq 1 + \left(\frac{1+\rho}{1+A-\delta}\right)^{1/\theta} \cdot (\psi_t - \overline{\psi})$ or, equivalently, $F(\psi_t) \simeq 2 - (1 + A - \delta)^{(\theta-1)/\theta}(1 + \rho)^{1/\theta} + \left(\frac{1+\rho}{1+A-\delta}\right)^{1/\theta} \psi_t$. Recovering the difference equation relating to the learning estimator, a two-equation system is derived,

$$\left\{ \begin{array}{l} \psi_t \simeq (1-\sigma_t)\psi_{t-1} \\ \quad + \sigma_t \left(\frac{1+A-\delta}{1+\rho}\right)^{1/\theta} \left[\frac{\psi_{t-1}}{z_{t-1}} - \left(2 - (1 + A - \delta)^{(\theta-1)/\theta}(1+\rho)^{1/\theta}\right)\right] \\ z_t = \psi_{t-1} \end{array} \right. \tag{7}$$

The first equation of system (7) is substantially different from the one in system (2); however, the local dynamic properties are exactly the same.

By linearizing system (7) in the neighborhood of the steady state, the same Jacobian matrix as in the original formulation is obtained and the trace and determinant of such matrix continue to be $Tr(J) = 1 - \overline{\sigma} + \overline{\sigma}/x$ and $Det(J) = \overline{\sigma}/x$ (with $x$ defined in the proof of proposition 2). Therefore, the system that is linearized in two steps has exactly the same local stability properties as the original one.

A conclusion identical to the previous one is withdrawn in the non-rival good growth model. Assume $F(\varphi_t) := \frac{E_t\varphi_{t+1}}{\varphi_t} = (1+A-\delta)^{(\theta-1)}(1+\rho)(1-\frac{1}{\varphi_t})$. Linearizing, $F(\varphi_t) \simeq 1 + (1 + A - \delta)^{(\theta-1)}(1 + \rho)\frac{1}{\overline{\varphi}^2}(\varphi_t - \overline{\varphi})$. The adaptive learning system comes,

$$\left\{ \begin{array}{l} \varphi_t \simeq (1-\sigma_t)\varphi_{t-1} \\ \quad + \sigma_t \frac{\left[(1+\rho)-(1+A-\delta)^{1-\theta}\right]^2}{(1+A-\delta)^{(1-\theta)}(1+\rho)} \left[\frac{\varphi_{t-1}}{v_{t-1}} - \frac{2(1+A-\delta)^{(1-\theta)}-(1+\rho)}{(1+A-\delta)^{(1-\theta)}}\right] \\ v_t = \varphi_{t-1} \end{array} \right. \tag{8}$$

Again, one finds relevant differences between the shape of the first equation of system (8) and the shape of the first equation of (4). Nevertheless, the coincidence in terms of local dynamic properties exists. As before, $Tr(J) = 1 - \overline{\sigma} + \overline{\sigma}/\widetilde{x}$ and $Det(J) = \overline{\sigma}/\widetilde{x}$, with $\widetilde{x}$ defined in section 2.2.

The one-sector endogenous growth model has allowed to illustrate that the moment of the linearization is irrelevant in what concerns local dynamic properties. This result is useful when analyzing higher dimensional systems, as the two-sector growth model of the next section.

### 3.3 A three-dimensional example

To understand how additional variables may be included into the analysis, consider now variables $X_t, Y_t \in \mathbb{R}$ and let $E_t X_{t+1} = B_t X_t$, with $B_t$ the adaptive learning estimator, and $Y_{t+1} = G(X_t, Y_t)$, $X_0, Y_0$ given. Adopting the same procedure as before, we define $F(X_t, Y_t) := E_t X_{t+1}/X_t$, and compute a linearized version of this function, $F(X_t, Y_t) \simeq 1 + F_X(\overline{X}, \overline{Y})(X_t - \overline{X}) + F_Y(\overline{X}, \overline{Y})(Y_t - \overline{Y})$. The adaptive learning rule leads to the following generic system,

$$\left\{ \begin{array}{l} Y_t = G(X_{t-1}, Y_{t-1}) \\ X_t = (1 - \sigma_t) \left[ X_{t-1} + \frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})} Y_{t-1} \right] \\ \quad + \frac{\sigma_t}{F_X(\overline{X},\overline{Y})} \left[ \frac{X_{t-1}}{Z_{t-1}} - \left(1 - F_X(\overline{X},\overline{Y})\overline{X} - F_Y(\overline{X},\overline{Y})\overline{Y}\right) \right] \\ \quad - \frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})} G(X_{t-1}, Y_{t-1}) \\ Z_t = X_{t-1} \end{array} \right. \tag{9}$$

The additional dimension does not allow to present, in simple terms, a stability condition as the one found for the two-dimensional system. However, through a numerical example, we show that the stability condition has the same basic characteristic already noticed: stability exists for a steady state gain sequence value below a given threshold.

The Jacobian matrix associated to the linearized system obtained from (9) is:

$$J = \begin{bmatrix} G_Y(\overline{X},\overline{Y}) & G_X(\overline{X},\overline{Y}) & 0 \\ (1-\overline{\sigma}-G_Y(\overline{X},\overline{Y}))\frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})} & 1-\overline{\sigma}+\frac{\overline{\sigma}}{F_X(\overline{X},\overline{Y})\overline{X}}-\frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})}G_X(\overline{X},\overline{Y}) & -\frac{\overline{\sigma}}{F_X(\overline{X},\overline{Y})\overline{X}} \\ 0 & 1 & 0 \end{bmatrix}$$

Trace, sum of principle minors and determinant of matrix $J$ are, respectively, $Tr(J) = 1 - \overline{\sigma} + \frac{\overline{\sigma}}{F_X(\overline{X},\overline{Y})\overline{X}} - \frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})} G_X(\overline{X},\overline{Y}) + G_Y(\overline{X},\overline{Y})$;

$\Sigma M(J) = (1-\overline{\sigma})\left(G_Y(\overline{X},\overline{Y}) - \frac{F_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})}G_X(\overline{X},\overline{Y})\right) + \frac{G_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})\overline{X}}\overline{\sigma} + \frac{\overline{\sigma}}{F_X(\overline{X},\overline{Y})\overline{X}}$ and $Det(J) = \frac{G_Y(\overline{X},\overline{Y})}{F_X(\overline{X},\overline{Y})\overline{X}}\overline{\sigma}$.

Stability conditions for three-dimensional systems are expressed as [see Brooks (2004)],

(i) $1 - Det(J) > 0$;

(ii) $1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0$;

(iii) $1 - Tr(J) + \Sigma M(J) - Det(J) > 0$;

(iv) $1 + Tr(J) + \Sigma M(J) + Det(J) > 0$.

Applying these conditions to the previously found trace, sum of principle minors and determinant implies reaching a not too straightforward stability result. A numerical example illustrates the obtainable results. Let $F_X(\overline{X},\overline{Y}) = 1, F_Y(\overline{X},\overline{Y}) = -0.5, G_X(\overline{X},\overline{Y}) = 0.25, G_Y(\overline{X},\overline{Y}) = 0.75$ and $\overline{X} = 0.1$. In this case, the linearized system under appreciation is:

$$\begin{bmatrix} Y_t - \overline{Y} \\ X_t - \overline{X} \\ Z_t - \overline{X} \end{bmatrix} = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.5\overline{\sigma} - 0.125 & 1.125 + 9\overline{\sigma} & -10\overline{\sigma} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} Y_{t-1} - \overline{Y} \\ X_{t-1} - \overline{X} \\ Z_{t-1} - \overline{X} \end{bmatrix}$$

From the Jacobian matrix, we obtain $Tr(J) = 1.875 + 9\overline{\sigma}$, $\Sigma M(J) = 0.875 + 16.625\overline{\sigma}$ and $Det(J) = 7.5\overline{\sigma}$. Applying the four stability conditions, one finds, respectively, (i) $\overline{\sigma} < 0.133$, (ii) $\overline{\sigma} < 0.071 \vee \overline{\sigma} > 0.157$, (iii) $\overline{\sigma} > 0$, (iv) $3.75 + 33.125\overline{\sigma} > 0$. The third and fourth conditions hold for any $\overline{\sigma} \in (0,1)$; the relevant stability condition is given by the intersection of the first two inequalities and, thus, the intended result is $\overline{\sigma} < 0.071$: the gain sequence must assume a steady state value below 0.071 in order to guarantee convergence to the steady state point $(\overline{X},\overline{Y})$.

# 4 Two-Sector Growth Model

To the intertemporal optimization problem addressed in section 2, we now add a second capital input (we consider only the rival final good case). Human capital is produced under a constant marginal returns technology, i.e., we consider a function $g((1-u_t)h_t)$, $g : \mathbb{R}_+ \to \mathbb{R}_+$, in which $h_t \geq 0$ represents the available stock of human capital and $1 - u_t \in (0,1)$ is the share of human capital allocated to the generation of additional quantities of this input. Conversely, $u_t$ will be the share of human capital used to produce final goods. The constant returns assumption implies $g' = B$, with $B$ a positive constant that will represent the productivity of the education sector.

The physical capital dynamic equation is similar to the one previously considered, with a relevant difference: the production function has two arguments, which are the two forms of capital, $f(k_t, u_t h_t)$, with $f : \mathbb{R}^2_+ \to$

$\mathbb{R}_+$. The tractability of the model requires considering a specific functional form for the production function; a Cobb-Douglas specification is assumed: $f(k_t, u_t h_t) = Ak_t^{\alpha}(u_t h_t)^{1-\alpha}$, with $\alpha \in (0,1)$ the output-capital elasticity. In this function, returns to scale are constant and marginal returns are diminishing.

Taking a same depreciation rate for both forms of capital, the two sector endogenous growth model is specified as follows,

$$\begin{aligned} Max \;\; V_0 &= \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t U(c_t) \\ \text{subject to} &: \\ k_{t+1} &= Ak_t^{\alpha}(u_t h_t)^{1-\alpha} - c_t + (1-\delta)k_t \\ h_{t+1} &= B(1-u_t)h_t + (1-\delta)h_t \\ & \quad k_0, h_0 \text{ given} \end{aligned}$$

In the above problem, the two capital inputs are the state variables. The process through which capital is generated is fully deterministic and assuming an efficient allocation of resources, then the levels of capital in $t+1$ are known with certainty. Consumption and the share of human capital allocated to each one of the possible uses are control variables and their dynamics depend on the expected values of the shadow-prices of capital. Therefore, we will not have a perfect knowledge about the values of $c_{t+1}$ and $u_{t+1}$; expectations are formed relating each one of these values.

## 4.1 The stability condition

To search for stability conditions in the two-sector competitive growth model under adaptive learning, we will need to assume a set of simplifying assumptions. The first assumption is that the representative agent does not choose an optimal path for the share $u_t$; the human capital share is assumed constant through time at the steady state level: $u_t = \overline{u}$. In this way, one can focus the attention on the consumption - physical capital ratio, as one did in previous sections. The second simplifying assumption consists in adopting the linearized estimator procedure of the last section; this is the only way through which it is possible to obtain explicit stability conditions under learning. The third shortcoming relates to the assumption of specific parameter values in order to obtain meaningful relations; otherwise, one would obtain cumbersome combinations of parameters that would give no relevant information about the dynamics of the system. Finally, a fourth assumption relates to the utility function: we restrict the analysis to the simplest case $U(c_t) = \ln c_t$.

The current-value Hamiltonian function of the two-sector problem takes the form

$$\begin{aligned} H(k_t, h_t, p_t, q_t, c_t, u_t) &= \ln c_t \\ &+E_t p_{t+1}\left[Ak_t^{\alpha}(u_t h_t)^{1-\alpha} - c_t - \delta k_t\right] \\ &+E_t q_{t+1}\left[B(1-u_t)h_t - \delta h_t\right] \end{aligned}$$

Variables $p_t$ and $q_t$ are the shadow-prices of physical capital and human capital, respectively.

Computing optimality conditions, the following set of relations is found:

$$H_c = 0 \Longrightarrow \frac{1}{c_t} = \left(\frac{1}{1+\rho}\right) E_t p_{t+1}$$

$$H_u = 0 \Longrightarrow \frac{E_t q_{t+1}}{E_t p_{t+1}} = (1-\alpha)\frac{A}{B}\left(\frac{k_t}{u_t h_t}\right)^{\alpha}$$

$$\begin{aligned} E_t p_{t+1} - p_t &= -H_k \Longrightarrow \\ \left(\frac{1}{1+\rho}\right)\left[1 + \alpha A\left(\frac{u_t h_t}{k_t}\right)^{1-\alpha} - \delta\right] E_t p_{t+1} &= p_t \end{aligned}$$

$$\begin{aligned} E_t q_{t+1} - q_t &= -H_h \Longrightarrow \\ \left(\frac{1}{1+\rho}\right)(1+B-\delta)E_t q_{t+1} &= q_t \end{aligned}$$

$$\lim_{t\to+\infty} k_t \left(\frac{1}{1+\rho}\right)^t p_t = \lim_{t\to+\infty} h_t \left(\frac{1}{1+\rho}\right)^t q_t = 0 \text{ (transversality condition)}$$

From the first and the third optimality conditions, an equation of motion for consumption is derived,

$$E_t c_{t+1} = \frac{1 + \alpha A\left(\frac{E_t u_{t+1} h_{t+1}}{k_{t+1}}\right)^{1-\alpha} - \delta}{1+\rho} c_t \tag{10}$$

For the proposed problem, the steady state is defined as the long run locus in which $u_t$ is constant and variables $k_t$, $h_t$ and $c_t$ grow at the same constant rate. This standard definition allows to present the steady state growth rate of the economy as $\gamma = B(1-\overline{u}) - \delta$. It also allows to define variables that assume constant long term values. The following are relevant: the consumption-physical capital ratio, $\psi_t := c_t/k_t$ (the expected value for this ratio will be $E_t\psi_{t+1} := E_t c_{t+1}/k_{t+1}$); the average product of capital, $\zeta_t := A\left(\frac{u_t h_t}{k_t}\right)^{1-\alpha}$ [under the assumption $u_t = \overline{u}$, the value of the defined variable in $t+1$ is known with certainty: $\zeta_{t+1} := A\left(\frac{\overline{u}h_{t+1}}{k_{t+1}}\right)^{1-\alpha}$]; and the ratio between co-state variables $Q_t := p_t/q_t$. This last ratio possesses a constant steady state value, as the previously defined variables, given the second optimality condition.

The shadow-prices equations of motion can be combined into a unique dynamic equation,

$$E_t Q_{t+1} = \frac{1+B-\delta}{1+\alpha\zeta_t-\delta} Q_t \tag{11}$$

In the steady state, $E_t Q_{t+1} = Q_t$ and, thus, $\overline{\zeta} = B/\alpha$. To obtain other steady state values, namely for $\psi_t$ and $u_t$, we resort once again to the stability conditions to write the relations

$$\zeta_{t+1} = \left(\frac{B(1-\overline{u})+1-\delta}{\zeta_t-\psi_t+1-\delta}\right)^{1-\alpha} \zeta_t \tag{12}$$

$$E_t\psi_{t+1} = \frac{1+\alpha\zeta_{t+1}-\delta}{(1+\rho)(\zeta_t-\psi_t+1-\delta)}\psi_t \tag{13}$$

Imposing $\overline{\zeta} := \zeta_{t+1} = \zeta_t$ and $\overline{\psi} := E_t\psi_{t+1} = \psi_t$, one finds the steady state values $\overline{u} = \frac{\rho}{1+\rho}\frac{1+B-\delta}{B}$ and $\overline{\psi} = \frac{\rho}{1+\rho}(1+B-\delta) + \frac{1-\alpha}{\alpha}B$. Observe that condition $\overline{u} < 1$ requires the constraint over parameters $B > \rho(1-\delta)$.

To analyze system (12)-(13) under learning, we will use the linearized estimator technique of the previous section, that is, we consider $F(\psi_t, \zeta_t) := \frac{1+\alpha\zeta_{t+1}-\delta}{(1+\rho)(\zeta_t-\psi_t+1-\delta)}$ and compute $b_t \simeq 1+F_\psi(\overline{\psi},\overline{\zeta})(\psi_t-\overline{\psi})+F_\zeta(\overline{\psi},\overline{\zeta})(\zeta_t-\overline{\zeta})$. The derivatives are $F_\psi(\overline{\psi},\overline{\zeta}) = \frac{(1-\alpha)\rho/\overline{u}+(1+\rho)}{1+B-\delta}$ and $F_\zeta(\overline{\psi},\overline{\zeta}) = -\frac{(1-\alpha)\rho/\overline{u}+(1+\rho)-\alpha}{1+B-\delta}$. The adaptive learning rule is the one used throughout the text, i.e., $b_t = b_{t-1} + \sigma_t\left(\frac{\psi_{t-1}}{\psi_{t-2}} - b_{t-1}\right)$, $b_0$ given. The learning system, then, will be

$$\begin{cases} \zeta_t = \left(\frac{B(1-\overline{u})+1-\delta}{\zeta_{t-1}-\psi_{t-1}+1-\delta}\right)^{1-\alpha}\zeta_{t-1} \\ \psi_t = (1-\sigma_t)\left[\psi_{t-1} + \frac{F_\zeta(\overline{\psi},\overline{\zeta})}{F_\psi(\overline{\psi},\overline{\zeta})}\zeta_{t-1}\right] \\ \quad + \frac{\sigma_t}{F_\psi(\overline{\psi},\overline{\zeta})}\left[\frac{\psi_{t-1}}{z_{t-1}} - \left(1 - F_\psi(\overline{\psi},\overline{\zeta})\overline{\psi} - F_\zeta(\overline{\psi},\overline{\zeta})\overline{\zeta}\right)\right] \\ \quad - \frac{F_\zeta(\overline{\psi},\overline{\zeta})}{F_\psi(\overline{\psi},\overline{\zeta})}\left(\frac{B(1-\overline{u})+1-\delta}{\zeta_{t-1}-\psi_{t-1}+1-\delta}\right)^{1-\alpha}\zeta_{t-1} \\ z_t = \psi_{t-1} \end{cases} \tag{14}$$

Proceeding as usual, the local linearization of (14) would allow to obtain a Jacobian matrix, relatively to which the analysis of stability is possible. However, an explicit stability result is not feasible to attain under so many involved combinations of parameters.

At this point, we introduce a vector of benchmark values for parameters, in order to explicitly present stability results, and in the next subsection we change some of these values in order to evaluate the impact of parameter changes over the stability outcome. We adopt, once again, the values used in Barro and Sala-i-Martin (1995), page 191, which are reasonable values to describe a developed economy (all values are per year): $\rho = 0.02$, $\delta = 0.05$, $\alpha = 0.5$, $B = 0.11$.

Given the steady state human capital share, the long run growth rate of the economy is $\gamma = \frac{B-\delta-\rho}{1+\rho}$ (the economy's growth rate benefits from a better human capital index of technology and it is penalized by higher depreciation and discount rates). In our specific numerical example, $\gamma = 0.0392$, i.e., the economy grows at an annual rate of 3.92%.

Parameter values allow for a full characterization of the long run locus; particularly, one may present the steady state values of the average product of capital, the consumption-capital ratio and of share $u_t$. The results are: $\overline{\zeta} = 0.22$; $\overline{\psi} = 0.131$; $\overline{u} = 0.189$. The specific example indicates that, in the long run, each unit of physical capital allows to produce 0.22 units of final goods, that consumption represents 13.1% of the stock of physical capital, and that 18.9% of the available human capital is allocated to the production of final goods, while the remaining 81.1% are used to generate additional human capital.

For the array of parameter values that was considered, we transform system (14) in the following system:

$$\begin{cases} \zeta_t = \sqrt{\frac{1.039}{\zeta_{t-1}-\psi_{t-1}+0.95}\zeta_{t-1}} \\ \psi_t = (1-\sigma_t)\psi_{t-1} - 0.534\zeta_{t-1} + \sigma_t \left[0.988\frac{\psi_{t-1}}{z_{t-1}} - 0.973\right] + 0.534\zeta_t \\ z_t = \psi_{t-1} \end{cases} \tag{15}$$

Linearizing around $(\overline{\zeta}, \overline{\psi})$,

$$\begin{bmatrix} \zeta_t - \overline{\zeta} \\ \psi_t - \overline{\psi} \\ z_t - \overline{\psi} \end{bmatrix} = \begin{bmatrix} 0.894 & 0.106 & 0 \\ -0.057 + 0.534\overline{\sigma} & 1.057 + 6.554\overline{\sigma} & -7.554\overline{\sigma} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \zeta_{t-1} - \overline{\zeta} \\ \psi_{t-1} - \overline{\psi} \\ z_{t-1} - \overline{\psi} \end{bmatrix} \tag{16}$$

Trace, sum of principle minors and determinant of the Jacobian matrix in (16), are straightforward to obtain: $Tr(J) = 1.951 + 6.554\overline{\sigma}$, $\Sigma M(J) = 0.951 + 13.358\overline{\sigma}$ and $Det(J) = 6.754\overline{\sigma}$. Applying the stability conditions for three dimensional discrete time systems: (i) $1 - Det(J) > 0 \Rightarrow \overline{\sigma} < 0.1480$; (ii) $1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0 \Rightarrow -0.2697 < \overline{\sigma} < 0.1351$; (iii) $1 - Tr(J) + \Sigma M(J) - Det(J) > 0 \Rightarrow \overline{\sigma} > 0$; (iv) $1 + Tr(J) + \Sigma M(J) + Det(J) > 0 \Rightarrow \overline{\sigma} > -0.1463$. Therefore, the stability condition for the two-sector endogenous growth model under the assumed benchmark set of parameter values is simply $0 < \overline{\sigma} < 0.1351$. One more time, the relevant condition for stability consists in guaranteeing a gain sequence steady state value below some determined amount, what is the same as saying that a minimum learning quality result must be satisfied.

### 4.2 Sensitivity analysis

To inquire about the robustness of the stability result obtained for the two-sector growth model, we now propose some changes on parameter values. We assume 25% changes (positive and negative) in each of the parameters $\rho$, $\delta$ and $B$. In each case, the other parameters maintain the values of the benchmark example. Table 2 presents the steady state growth rate and the steady state value of each assumed variable.

| | $\gamma$ | $\overline{\zeta}$ | $\overline{\psi}$ | $\overline{u}$ |
|---|---|---|---|---|
| $\rho = 0.025$ | 0.0341 | 0.22 | 0.1358 | 0.2350 |
| $\rho = 0.015$ | 0.0443 | 0.22 | 0.1257 | 0.1424 |
| $\delta = 0.0625$ | 0.0270 | 0.22 | 0.1354 | 0.1867 |
| $\delta = 0.0375$ | 0.0515 | 0.22 | 0.1310 | 0.1912 |
| $B = 0.1375$ | 0.0662 | 0.275 | 0.1588 | 0.1551 |
| $B = 0.0825$ | 0.0122 | 0.165 | 0.1027 | 0.2453 |

Table 2 - Steady state values on the two-sector growth model.

As the generic results have pointed out, the higher are the discount rate and the depreciation rate, the lower will be the growth rate of the economy and the higher will be the steady state value of the consumption-capital ratio. The share of human capital allocated to the production of final goods falls with an increase in the depreciation rate, but increases with a higher discount rate. Relatively to parameter $B$, the higher is its value, the faster the economy will grow in the long run. The average product of capital and the consumption-capital ratio also benefit from an improved technology level in the education sector. The better is the technology, the larger will be the share of human capital allocated to educative activities relatively to the production of final goods.

Proceeding as previously, i.e., by constructing the adaptive learning system and by linearizing it in the steady state neighborhood, we compute trace, sum of principle minors and determinant; this allows to evaluate each one of the four imposed stability conditions. Table 3 presents the admissible values for the steady state gain sequence, for each one of the conditions (i), (ii), (iii) and (iv), that appear at the end of the previous subsection.

| | $(i)$ | $(ii)$ | $(iii)$ | $(iv)$ |
|---|---|---|---|---|
| $\rho = 0.025$ | $\overline{\sigma} > 0.1546$ | $-0.2342 < \overline{\sigma} < 0.1415$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.1532$ |
| $\rho = 0.015$ | $\overline{\sigma} > 0.1415$ | $-0.3228 < \overline{\sigma} < 0.1288$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.1394$ |
| $\delta = 0.0625$ | $\overline{\sigma} > 0.1498$ | $-0.2745 < \overline{\sigma} < 0.1366$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.1480$ |
| $\delta = 0.0375$ | $\overline{\sigma} > 0.1463$ | $-0.2650 < \overline{\sigma} < 0.1337$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.14462$ |
| $B = 0.1375$ | $\overline{\sigma} > 0.1818$ | $-0.3588 < \overline{\sigma} < 0.1622$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.1790$ |
| $B = 0.0825$ | $\overline{\sigma} > 0.1149$ | $-0.1804 < \overline{\sigma} < 0.1074$ | $\overline{\sigma} > 0$ | $\overline{\sigma} > -0.1141$ |

Table 3 - Stability conditions in the two-sector growth model, under different parameter values.

Table 3 clearly establishes a pattern: it is the upper bound of the second stability condition that in every case determines the stability result, and this always corresponds to a given upper limit that is necessary to impose to the steady state gain sequence value. Conditions (iii) and (iv) are true conditions under $\overline{\sigma} \in (0, 1)$, while condition (i) requires, in every circumstance, a value for $\overline{\sigma}$ above the constraint that is set by the second condition.

Figures 2, 3, and 4 illustrate the stability results; each graphic corresponds to the relation between each one of the parameters $\rho$, $\delta$ and $B$ and the steady state value of the gain sequence. In every case, the line corresponding to stability condition (i) is the one located above all the others; the second line from above is the upper bound on the second stability condition, and it is the line that effectively translates the border between stability and instability; the two lines located below zero are, respectively, the ones corresponding to condition (iv) and to the lower bound of (ii).

The figures (and table 3) also allow to perceive another relevant pattern: the higher are the values of any of the three parameters, the discount rate, the depreciation rate and the education productivity level, the larger is the value of the boundary on the gain sequence value. This means that the learning effort required for long run stability diminishes with increases on the values of any of the assumed parameters. A larger $B$ relaxes the learning constraint and allows the economy to grow faster in the steady state. Larger discount and depreciation rates also allow to attain stability with less learning quality requirements, but these penalize growth.

*** *figures 2, 3, 4* ***

As in section 2, one may complete the analysis by considering the effects of learning costs. Consider, in this case, that learning relating to the improvement of the performance of expectations requires resources withdrawn from the education sector, such that the equation of motion concerning the human capital variable is now $h_{t+1} = [B(1 - \overline{u}) - \upsilon]h_t + (1 - \delta)h_t$, with $\upsilon \in (0, 1)$ the share of human capital dedicated to improve learning in order to better forecast future consumption levels.

The steady state growth rate of the economy becomes $\gamma = B(1-\overline{u})-\upsilon-\delta$. Steady state ratios are $\overline{\zeta} = (B - \upsilon)/\alpha$ (the marginal product of capital falls with an increase in the learning costs); $\overline{\psi} = \frac{\rho}{1+\rho}(1+B-\upsilon-\delta)+\frac{1-\alpha}{\alpha}(B-\upsilon)$ (in the steady state, a higher value of the learning parameter implies a fall on the level of consumption relatively to the capital stock); finally, $\overline{u} = \frac{\rho}{1+\rho}\frac{1+B-\upsilon-\delta}{B}$ (the larger are the learning costs, the lower will be the share of human capital allocated in the steady state to the production of final goods). In the present case, $\overline{u} \in (0, 1)$ requires $-(1-\upsilon-\delta) < B < \rho(1-\upsilon-\delta)$. The growth rate is, then, presentable as $\gamma = \frac{B-\rho-\upsilon-\delta}{1+\rho}$. Effectively, one observes that the higher is the level of the human capital resources diverted to learning, the lower will be the economy's growth rate.

Once again, the trade-off is evident: a larger $\upsilon$ share penalizes economic growth, the average productivity of physical capital and the relative level of steady state consumption. Nevertheless, a too low value of $\upsilon$ may imply that the representative agent is unable to learn enough to guarantee convergence to the steady state.

As in section 2, we assume the following gain sequence function: $\overline{\sigma}(\upsilon) = \widehat{s}(1-\upsilon)^2$, with $\widehat{s}$ a positive parameter lower than 1. According to this function specification, the higher is the value of $\upsilon$, the more the agent learns (the lower is $\overline{\sigma}$): $\overline{\sigma}_\upsilon < 0$. Let us resort to a numerical example. Consider the benchmark parameter values used to evaluate the two-sector growth model and assume $\widehat{s} = 0.14$. Assume, as well, the alternative values $\upsilon = 0, 0.01, ..., 0.04$. Table 4 indicates the obtained results.

| $\upsilon$ | $\gamma$ | $\overline{\zeta}$ | $\overline{\psi}$ | $\overline{u}$ | $\overline{\sigma}$ | Stability condition | Stability |
|---|---|---|---|---|---|---|---|
| 0 | 0.0392 | 0.22 | 0.1308 | 0.1889 | 0.14 | $\overline{\sigma}<$ 0.1351 | *No* |
| 0.01 | 0.0294 | 0.2 | 0.1206 | 0.1872 | 0.1372 | $\overline{\sigma}<$ 0.1251 | *No* |
| 0.02 | 0.0196 | 0.18 | 0.1104 | 0.1854 | 0.1345 | $\overline{\sigma}<$ 0.1150 | *No* |
| 0.03 | 0.0098 | 0.16 | 0.1002 | 0.1836 | 0.1317 | $\overline{\sigma}<$ 0.1048 | *No* |
| 0.04 | 0 | 0.14 | 0.09 | 0.1818 | 0.1290 | $\overline{\sigma}<$ 0.0945 | *No* |

Table 4 - Stability result with learning costs: the two-sector growth model.

The main difference relatively to the one-sector case analyzed in section 2 is that in the present circumstance the stability condition depends on the value of the learning costs parameter. Thus, to build the table, one has to recompute derivatives $F_\psi(\overline{\psi}, \overline{\zeta})$ and $F_\zeta(\overline{\psi}, \overline{\zeta})$; their values are $F_\psi(\overline{\psi}, \overline{\zeta}) = \frac{(1-\alpha)\frac{\rho}{\overline{u}}\frac{B-\upsilon}{B}+(1+\rho)}{1+B-\upsilon-\delta}$ and $F_\zeta(\overline{\psi}, \overline{\zeta}) = -\frac{(1-\alpha)\frac{\rho}{\overline{u}}\frac{B-\upsilon}{B}+(1+\rho)-\alpha}{1+B-\upsilon-\delta}$. Note, as well, that $\left.\frac{\partial\zeta_t}{\partial\psi_{t-1}}\right|_{(\overline{\psi},\overline{\zeta})} = \frac{1-\alpha}{\alpha}\frac{\rho}{\overline{u}}\frac{B-\upsilon}{B}$ and $\left.\frac{\partial\zeta_t}{\partial\zeta_{t-1}}\right|_{(\overline{\psi},\overline{\zeta})} = 1 - \frac{1-\alpha}{\alpha}\frac{\rho}{\overline{u}}\frac{B-\upsilon}{B}$. The relevant stability condition is, in any case, the upper bound of the second condition that relates trace, sum of principle minors and determinant of the Jacobian matrix. With $\upsilon = 0$, the stability result is the same as in the benchmark case we have analyzed. We have chosen a value for $\widehat{s}$ that does not allow for stability when no investment in learning is undertaken.

The relevant conclusion in this case is that a higher $\upsilon$ allows for a better steady state learning result, while implying a worse steady state performance (lower growth, lower average product of capital, lower relative consumption level and a poorer allocation of human capital to the production of final goods). Nevertheless, as the gain sequence outcome is improved through a larger resource allocation to learning, also the stability condition changes, becoming increasingly demanding; because as $\upsilon$ rises, the stability requirement changes faster than the steady state of the gain sequence, investing in learning through the use of human capital does not allow to modify the non stability result.

A different result would be obtained if the human capital resources needed to learn were non-rival, i.e., if by accumulating human capital, the representative agent becomes better prepared to forecast optimal future consumption, without the need for diverting resources from one activity to the other. In such case, the agent could maintain the same economy's growth rate and the other steady state results while improving the quality of learning. This scenario would imply an unchangeable stability condition, and therefore the higher the level of accumulated human capital, the more likely would be to assure stability, without the need for reallocating resources.

# 5 Conclusion

We have analyzed an endogenous growth model under the conventional intertemporal utility maximization setup. The future value of the control variable (consumption) is not known with certainty in the present moment. The way expectations about the next period value of this variable are formed is the central point of the analysis. Abandoning the perfect foresight assumption and taking into account an adaptive learning rule, we have studied stability in both one-sector and two-sector environments.

The learning mechanism does not modify the long term steady state results (unless one considers that learning is costly, and therefore diverting resources to improve the formation of expectations lowers the long run economy's growth rate). However, it introduces relevant changes in what concerns the stability properties of the steady state. There is a relevant common result for the various models one has addressed: local stability requires a minimum learning efficiency. If the quality of learning is low (the gain sequence steady state value is relatively high), the economy diverges from the steady state characterized by a constant growth rate and a constant consumption-capital ratio.

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## Figures

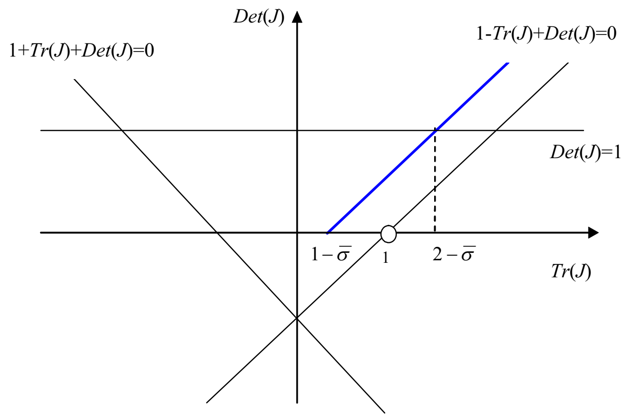


Figure 1: Trace-determinant diagram in the AK growth model with a rival final good.

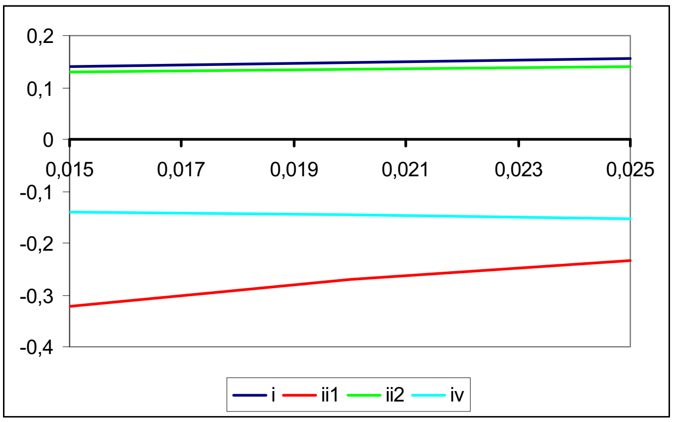


Figure 2: Stability relation between $\rho$ and $\overline{\sigma}$.

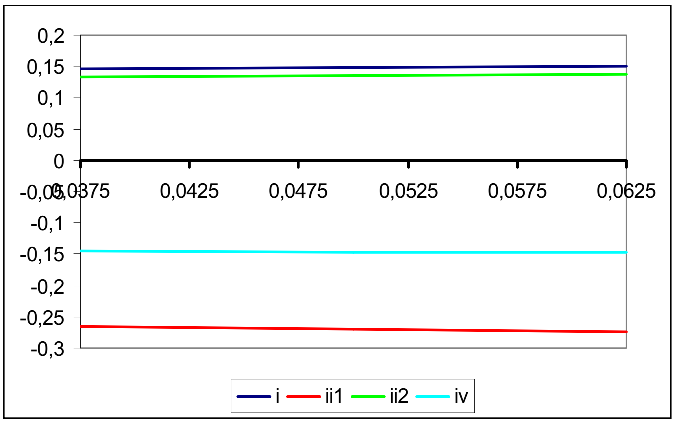


Figure 3: Stability relation between $\delta$ and $\overline{\sigma}$.

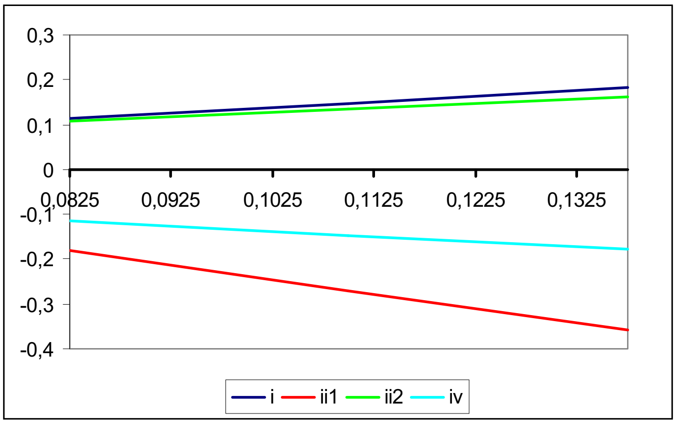


Figure 4: Stability relation between $B$ and $\overline{\sigma}$.