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Abstract

In this paper we apply the techniques of symbolic dynamics and chaos control to the analysis of a labor market model which shows chaotic behavior and large volatility in employment flows. The possibility that chaotic dynamics may arise in modern labor markets had been totally strange to economics until recently. In an interesting paper Bhattacharya and Bunzel [2] have found that the discrete time version of the Pissarides-Mortensen matching model, as formulated in Ljungqvist and Sargent [23], can easily lead to chaotic dynamics under standard sets of parameter values. This paper explores this version of the model with two main objectives in mind: (i) to clarify some open questions raised by Bhattacharya and Bunzel by providing a rigorous proof of the existence of chaotic dynamics in the model; and (ii) to show that this type of dynamics can be easily controlled by linear feedback techniques — the OGY method — without producing modifications to the original model, apart from locally changing its type of stability. These techniques may be of significant importance for the study of economic theory and policy, in particular, if complexity becomes more frequently encountered in the models developed to properly describe the behavior of modern economies, and the view of purely exogenous shocks as explaining cycles and volatility looses its large predominance in contemporary economics.

Keywords: Symbolic Dynamics, Chaos Control, Matching and Unemployment

1 Introduction

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‡We deeply regret the death of José Sousa Ramos on December 2006. José was a member of the Department of Mathematics, IST, Technical University of Lisbon, Lisbon, Portugal.

In a recent paper, Levin [22] discusses the set of results presented over the last decade by various prominent physicists which led to the conclusion that black holes seem to be susceptible to chaos. Levin argues that the most realistic description available of a spinning pair of black holes is chaotic motion, and goes on to complain that in physics and cosmology "chaos has not received the attention it deserves in part because the systems studied have been highly idealized". In contrast, in economics we have the interesting fact that even some of the most simple and highly idealized models describing modern economies can easily lead to chaotic dynamics.[1]

In this paper we apply the techniques of symbolic dynamics and chaos control to the analysis of a labor market model which shows chaotic behavior and large volatility in employment flows. The possibility that chaotic dynamics may arise in modern labor markets had been totally strange to economics until recently, at least as far as we are aware of. However, in an interesting paper Bhattacharya and Bunzel [2] have found that the discrete time version of the Pissarides-Mortensen matching model, as formulated in [23], can easily lead to chaotic dynamics under standard sets of parameter values. This paper explores this version of the model having two basic objectives in mind: (i) to clarify some open questions raised in the paper of Bhattacharya and Bunzel by providing a rigorous proof of the existence of chaotic dynamics in the model; and (ii) to show that this type of dynamics can be easily controlled by linear feedback techniques — the OGY method — without producing modifications to the original model, apart from locally changing its type of stability. These two techniques may be of significant importance for the study of economic theory and policy, in particular, if complexity becomes more frequently encountered in the models developed to properly describe the behavior of modern economies, and the view of purely exogenous shocks as explaining cycles and volatility looses its large predominance in contemporary economics.

The paper is organized as follows. In section 2 we present some basic facts of modern labour markets and the fundamental aspects of the discrete time version of the matching model as developed in [23]. In section 3, the dynamics of the matching model is studied in great detail, including stable steady states, periodic motion, bifurcations and chaos. Particular emphasis will be given to the application of symbolic dynamics to provide a rigorous proof of chaos in this model. Section 4 deals with the control of chaotic motion. Section 5 concludes.

2 Basic Facts and the Matching Model

Until recently, the two largely dominant views of unemployment were those of the Classical model of macroeconomic behavior and the Keynesian view of sticky wages due to various forms of market frictions (imperfect information and market power). In the former model, the labor market works as a permanent auction process with fully informed and rational agents and, under these assumptions, any workers remaining unemployed for a significant period of time are entirely

[1] See, e.g., [36], [30], [35], [34], [3], [17], [6] and [7]. The potential for very complex behavior significantly increases if models become somewhat less reductionist, e.g., if heterogenous agents and different learning processes are also taken into account, [18], [8], [12], [9], [15].

the result of a rational response to optimal choices of leisure/work and consumption. In contrast to this view of voluntary unemployment, in the Keynesian model unemployment was the involuntary result of the extreme difficulty to move prices and wages quickly enough to accommodate shocks in aggregate demand and aggregate supply (technological shocks).

These two views were seriously challenged over the last two decades. In a succession of papers, various economists developed what has become known as the *search and matching theory of unemployment*, where creation and destruction of jobs become the major channel to explain the level of unemployment over time.[2] The evidence is very favorable to the basic tenets of this theory, not only due to the fact that the flows *in* and *out* of employment are (unexpectedly) massive in modern societies — see, e.g., [14] and [5] — but also because the econometric results concerning aggregate functions associated with this theory have been positive, as recently emphasized in [32] for the matching function.

Some evidence of two major points of this explanation of unemployment is presented in Figure 1: job flows IN and OUT of employment as a percentage of working age population, and the ratio of the rate of job vacancies to the rate of unemployment for the US economy.[3] In panel (a) we can observe that over the 1970s/1980s, on average, around 1.85% of the total working age population was moving into employment (IN), against around 1.8% moving OUT, *in every month*. Taking into account that the working age population of the US was on average above 100 million people over that period, we obtain the interesting finding that in every month a number close to 2 million people were getting employed and a slightly lower number were moving into unemployment. The significant complexity of these huge flows is illustrated in the two following panels: (b) presents a cross plot of the two flows and suggests the form of an attractor that is not totally strange to those obtained in some forms of chaotic dynamics; and in (c) the *net* flows of creation and destruction of jobs (IN−OUT) shows a time series that easily resembles either a linear stochastic process or a chaotic one. Finally, panel (d) presents evidence of a measure of the market tightness (the ratio of the rate of vacancies to the rate of unemployment), which also shows an highly irregular pattern over time.

Why would the labor market of one of the most developed and sophisticated economies in the world we live in behave in such a volatile way? One possibility, and usually the most favoured one in the dominant view of economics, is that the economy has an inherently linear structure and is hit by permanent exogenous shocks. As these shocks are entirely unpredictable, they render the dynamics and the cycles hardly predictable and controllable. Another more recent view, and one which we consider more consistent and realistic, is based on the possibility that the economy has a structure that is nonlinear and the cycles are a manifestation of this characteristic, either with or without external shocks. In what follows, a very simple and fully deterministic model will be presented that is capable of explaining such type of volatility with standard parameter sets.

Let us assume that in every period of time there are large flows of workers

[2] Seminal papers in this field are [25], [24], [16], [29] and [33]. On the empirical level see [14], [4] and [5] for the US economy, while [10] and [13] present favourable evidence for the European economies.

[3] The data was colected by [5].

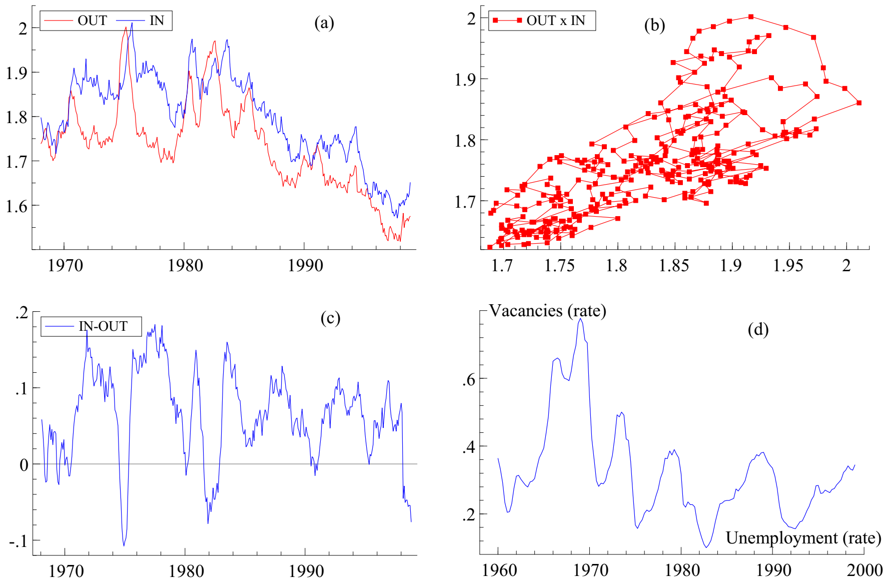

Figure 1: *Complex behvior in the US labor market: 1960–2000.*

moving into and out of employment: a certain number of job vacancies is posted by firms (v_t) and there is a total measure of workers looking for jobs (u_t).[4] When a worker and a firm reach an agreement there is a successful match, and the total number of these matches is given by the aggregate matching function

$$M\left(u_t, v_t\right) = A u_t^{\alpha} v_t^{1-\alpha} \tag{1}$$

$A > 0,\ \alpha \in (0, 1)$. The intuition behind (1) is very simple: the higher is u, the easier it will be for firms to get a worker with the desired qualifications; and the higher is the level of vacancies posted by firms v, the higher is the probability that a worker will find an appropriate job. For simplicity we will assume A as a constant. However, a more adequate treatment would consist of treating A as a variable dependent on the level of public provision of information by public agencies with the objective of increasing the number of successful matches.[5]

The measure of labor tightness is given by the ratio

$$\theta_t \equiv \frac{v_t}{u_t}$$

[4] Notice that, in fact, the variables v and u are in their nature very similar to the aggregates above described by IN and OUT.

[5] If public information is costly, an optimal level of this kind of asset would have to be determined endogenously in this model. However, this procedure would add no significant novelty to the main objective of the paper (to provide a rigorous proof of chaos in the model and to exemplify the application of chaos control techniques), and for this reason we keep it as simple as possible.

Then, the probability of a vacancy being filled at t is given by

$$q\left(\theta_t\right) \equiv \frac{M\left(u_t, v_t\right)}{v_t} = A\theta_t^{-\alpha}.$$

Let n_{t+1} be the total number of employed workers at the beginning of $t+1$ and let s be defined as the probability of a match being dissolved at t. Therefore we have

$$n_{t+1} = (1-s)\,n_t + q\left(\theta_t\right) v_t,$$

where $\theta_t \equiv v_t/u_t = v_t/(1-n_t)$. Notice that $(1-s)\,n_t$ gives the number of undissolved matches prevailing at t and passed on to $t+1$, while $q\left(\theta_t\right) v_t$ represents the number of new matches formed at t with the available number of unemployed workers and vacancies.

As shown in [23], the model can be solved for the decentralized outcome of a Nash bargaining game between workers and firms. Nevertheless, as we are interested not only on the modelling side of economic chaotic dynamics but also on the normative side (which is related to the behavior of government and to the control of chaos in this case), we should focus upon the central planner solution to the matching model. The objective function of the central planner is given by

$$U(n,v) = \phi n_t + z\left(1-n_t\right) - cv_t$$

where ϕ, z and c are parameters that represent, respectively, the productivity of each worker, the lost value of leisure due to labor effort, and the cost that firms incur per vacancy placed in the market.[6]

Therefore, the planner chooses v_t and the next period's employment level, n_{t+1}, by solving the following dynamic optimization problem

$$\max_{v_t, n_{t+1}} \sum_{t=0}^{\infty} \beta^t \left[\phi n_t + z\left(1-n_t\right) - cv_t\right]$$

subject to

$$n_{t+1} = (1-s)\,n_t + q\left(\frac{v_t}{1-n_t}\right) v_t,$$

where β is the time discount rate and an initial condition n_0 is given. The Lagrangian can be written as [7]

$$L = \sum_{t=0}^{\infty} \left\{ \beta^t \left[\phi n_t + z\left(1-n_t\right) - cv_t\right] + \lambda_t \left[(1-s)\,n_t + q\left(\frac{v_t}{1-n_t}\right) v - n_{t+1} \right] \right\}.$$

The first order conditions (FOC), for an interior solution, are given by

$$\frac{\partial L}{\partial v_t} = -\beta^t c + \lambda_t \left[q'\left(\theta_t\right)\theta_t + q\left(\theta_t\right)\right] = 0$$

$$\frac{\partial L}{\partial n_{t+1}} = -\lambda_t + \beta^{t+1}\left(\phi - z\right) + \lambda_{t+1}\left[(1-s) + q'\left(\theta_{t+1}\right)\theta_{t+1}^2\right] = 0.$$

[6] Notice the trade–off between vacancies and unemployment in this objective function. The first right hand term represents the benefits to society from successful matches (working), while the last two give the leisure costs and the costs associated with posting vacancies.

[7] See [11] for a detailed treatment of this standard procedure in dynamic economic analysis.

The very interesting point in [2] was the manipulation of these FOC to arrive at a reduced equation that can easily lead to chaotic dynamics. From the first FOC we get

$$\lambda_t = \frac{\beta^t c}{q'(\theta_t)\,\theta_t + q(\theta_t)}$$

and substituting this and the corresponding expression for λ_{t+1} into the second FOC we obtain

$$a\theta_{t+1}^{\alpha} - b\theta_{t+1} = \theta_t^{\alpha} - d \tag{2}$$

where $\alpha \in (0,1)\,,\, a \equiv \beta(1-s) \in (0,1)\,,\, b \equiv A\alpha\beta > 0,\, d \equiv (A/c)(1-\alpha)(\phi - z) > 0.$

Equation (2) gives the law of motion for the index of labor market tightness in the economy under the planner's solution. In other words, given an initial condition θ_0, equation (2) completely characterizes the trajectory of θ and the whole economy. So, the backward dynamics of this model can be characterized by the four-parameter family of maps $g : [0, g_{\max}] \to [0, g_{\max}]$, where

$$g(\theta) = (a\theta^{\alpha} - b\theta + d)^{\frac{1}{\alpha}}\,,$$

$\alpha \in (0,1)\,,\ a \in (0,1)\,,\ b, d > 0$ and

$$\theta_{\max} = \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$$

where $g_{\max}$ is implicitly defined as the lowest positive root of the equation

$$a g_{\max}^{\alpha} - b g_{\max} + d = 0.$$

The first derivative of the map g can be calculated as

$$g'(\theta) = (a\theta^{\alpha} - b\theta + d)^{\frac{1-\alpha}{\alpha}} \left(a\theta^{\alpha-1} - \frac{b}{\alpha}\right),\ \theta \in [0, g_{\max}]\,,$$

which implies that g is unimodal with a unique maximum (critical point) at $\theta_{\max} = \left(\frac{\alpha a}{b}\right)^{\frac{1}{1-\alpha}}$. In addition, g has a unique fixed point located to the right of $\theta_{\max}$ if $g(\theta_{\max}) > \theta_{\max}$.

The unique fixed point of g is denoted by θ_* and is implicitly given by

$$a\theta_*^{\alpha} - b\theta_* = \theta_*^{\alpha} - d. \tag{3}$$

Despite the impossibility of the computation of an explicit solution for θ_*, the unicity of this solution is obvious by considering

$$f_1(\theta) = a\theta_*^{\alpha} - b\theta_* \quad \text{and} \quad f_2(\theta) = \theta_*^{\alpha} - d$$

where $f_1(\theta)$ is monotonically decreasing for θ from $\theta_{\max}$ to $+\infty$ and $f_2(\theta)$ is monotonically increasing for θ from 0 to $+\infty$. Therefore $f_1(\theta) = f_2(\theta)$ has a unique solution for $\theta > \theta_{\max}$ and this is illustrated in *Figure 2.*

The fixed point is an attractor in the case of backward dynamics if $|g'(\theta_*)| < 1$ and in forward dynamics if $g'(\theta_*) < -1$. For $g'(\theta_*) = -1$ a period-doubling bifurcation occurs and the fixed point changes stability. Since it is not possible to obtain a closed form expression for θ_*, this condition cannot be checked in general but can be checked for each set of parameters separately.

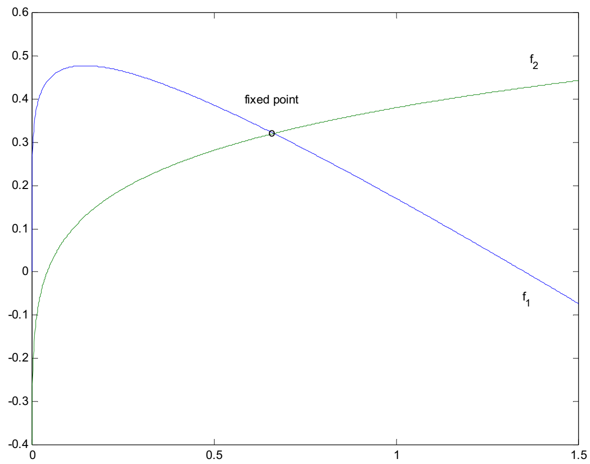

Figure 2: *Unicity of the fixed point.*

3 Chaotic Dynamics in the Matching Model

Bhattacharya and Bunzel [2] suggest three examples for the dynamics of the map g. In the first case a period 3-cycle is found, which implies the existence of chaos in the Li-Yorke sense if the Sharkovsky order is applied. In the second example, for a suitable choice of parameters, the authors do not find a period three orbit but show the existence of chaos by applying Mitra's sufficient condition for chaos in unimodal maps.[8] Finally, in the third example, no period three orbit is found and also the sufficient condition of Mitra is not verified. In this case, the very existence of chaos for the unimodal map is questioned on the grounds of a lack of logical proof of such dynamics.

In what follows we revisit these examples and add some further information, hoping to contribute to the clarification of some open questions raised by the interesting paper of Bhattacharya and Bunzel. For this purpose, a symbolic dynamics approach is developed for the unimodal map g, which allow us to perform the computation of the topological entropy for any choice of parameters, and, of course, permit us to classify the complexity of the map since positive topological entropy implies chaotic dynamics.

Example 1: For $a = 0.961863$, $b = 0.947099$, $d = 0.458566$, $\alpha = 0.155693$ there is a unique steady state $\theta_* = 0.4486$, and $g'(\theta_*) = -2.13$, indicating that θ_* is locally unstable in the case of backward dynamics and stable in normal forward dynamics. Two 3-cycles are found by solving the nonlinear equation $g^3(\theta) = \theta$, which are: $\{0.1122, 1.2591, 0.00018\}$ and $\{0.00051, 0.1624, 1.2054\}$. Having established these results for that parameter specification, the map g

[8] As argued in [28], for a continuous unimodal map $f : X \to X$, where X is a non-negative interval, $x_{\max}$ is the critical point such that $f(x_{\max}) > x_{\max}$ and x_* is the unique fixed point of the map such that $x_* > x_{\max}$, Mitra states the following: If f satisfies $f^2(x_{\max}) < x_{\max}$ and $f^3(x_{\max}) < x_*$, then (X, f) exhibits topological chaos.

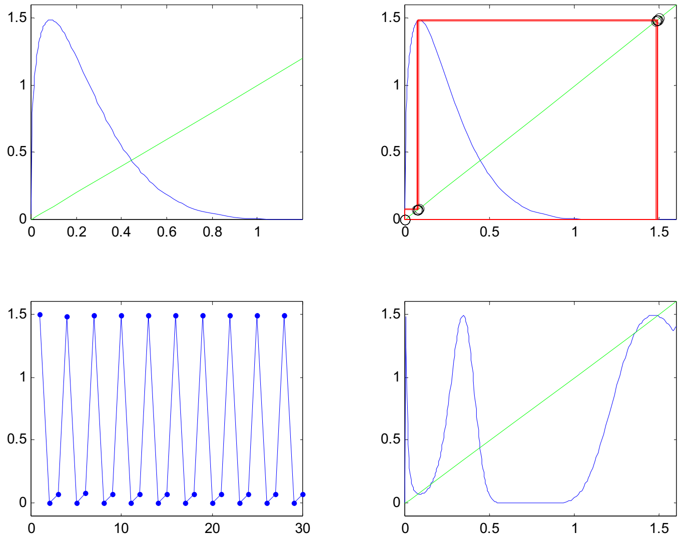

Figure 3: *Existence of a 3-cycle.*

has a 3-cycle, and the existence of all other periods follows from Sharkovsky's Theorem. The Li-Yorke Theorem then establishes that the existence of a 3-cycle implies the existence of chaotic equilibria.

Example 2: In this example, as argued by [2], a 3-cycle could not be found for the following parameter values: $a = 0.75$, $b = 0.54$, $d = 0.62$, $\alpha = 0.15$. This is normal because if we fix a, d, α and vary b, for these values we have some chaotic orbits, and the period three orbit appears *only* for $b = 0.9$. Figure 3 shows the map g, the cobweb and the time series for the period 3-orbit and g^3.

For a more clear perspective on the type of dynamics of this example, Figure 4 presents the bifurcation diagram and the Lyapunov exponent when the parameter b is varying between 0.2 and 1.0. We can observe a period-doubling route to chaos and stability windows for the 3-cycle and 5-cycle where the Lyapunov exponent is negative. We can also observe that for $b = 0.54$ the Lyapunov exponent is positive which is a sufficient condition for chaos. Moreover, if a map g possesses a periodic point of period not equal to a power of two then the dynamical system is complex, which is a well-known fact from Sharkovsky's work [37]. In particular, there are cycles of arbitrarily large periods, homoclinic trajectories, chaos in the sense of Li-Yorke, positive topological entropy, and so on. From the bifurcation diagram in Figure 4 we observe the existence of a period 5 orbit for $b = 0.58$, which implies that orbits of all periods exist and in consequence chaos exists in the Li-Yorke sense.

Example 3: The set of parameters for this example are: $a = 0.9$, $b = 0.7$, $d = 0.6$, $\alpha = 0.2$. In [2] it is argued that for these values a 3-cycle can not be found, the sufficient condition of Mitra is not satisfied, and the possibility

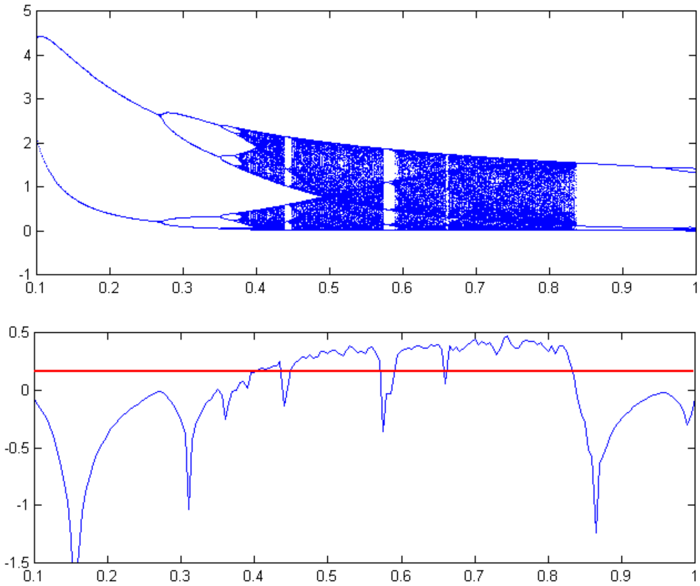

Figure 4: *Bifurcation diagram and variation of the Lyapunov exponent.*

of chaotic equilibrium dynamics is questioned. Figure 5 shows the bifurcation diagram for a, d, α fixed and for b varying from 0.1 to 1.3. It is straightforward to see in panel (a) that a period three orbit occurs for $b = 1.2$, and the unimodal map is in transition to chaos when $b = 0.7$, the 3-cycle appearing at the end of the transition as indicated by the Sharkovsky order. This should not be interpreted as suggesting that a chaotic equilibrium is not verified for $b = 0.7$, actually a two-piece chaotic attractor exists for this value as shown in panel (b).

In order to clarify whether there are chaotic dynamics or not under certain ranges of parameters values, we suggest that a bifurcation diagram, the variation of the Lyapunov exponent, the existence of a periodic point of period not equal to a power of two, and symbolic dynamics are some techniques that can give a clear answer to this problem. Figure 5 shows that for $b = 0.4$ the map is not chaotic, but starting with some bifurcation value for b the equilibrium moves into chaotic motion.

Turning to symbolic dynamics, we consider again the unimodal map $g : [0, g_{\max}] \to [0, g_{\max}]$. This kind of map has symbolic dynamics relative to a partition at the critical point $\theta_{\max}$. This is illustrated in Figure 6 for the parameter values presented in Example 2. So, any numerical trajectory $\theta_0, \theta_1, \theta_2, ...$ for the map g corresponds to a symbolic sequence

$$\sigma(\theta_0) = \sigma_0(\theta_0)\,\sigma_1(\theta_0)\,\sigma_2(\theta_0)\,... = \sigma_0\sigma_1\sigma_2...$$

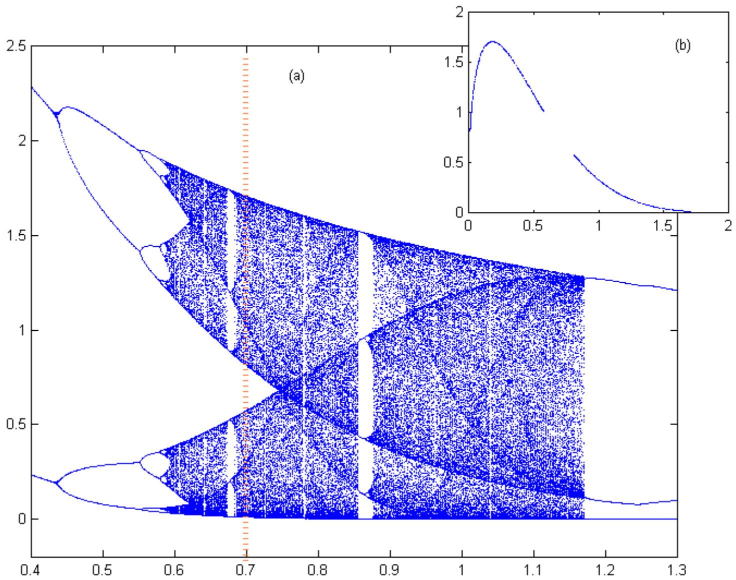

Figure 5: *(a) Bifurcation diagram when b is varied between* 0.4 *and* 1.3*, maintaing $a = 0.9$, $d = 0.6$, $\alpha = 0.2$. (b) Two-piece attractor for $b = 0.7$*

where $\sigma_i \in \{L, C, R\}$ depending on where the point θ_i falls in, i.e.,

$$\sigma_i(\theta_0) = \begin{cases} L & \text{if } g^i(\theta_0) < \theta_{\max} \\ C & \text{if } g^i(\theta_0) = \theta_{\max} \\ R & \text{if } g^i(\theta_0) > \theta_{\max} \end{cases} .$$

All symbolic sequences made of these letters may be ordered in the following way. First, there is a natural order

$$L < C < R. \tag{4}$$

Next, if two symbolic sequences σ and σ' have a common leading string σ^*, i.e.,

$$\sigma = \sigma^* s_i..., \quad \sigma' = \sigma^* t_i..., \quad s_i \neq t_i$$

then they must be ordered according to (4). The ordering rule is: if σ^* is even, i.e., if contains an even number of R, the order of σ and σ' is given by that of s_i and t_i, and if σ^* is odd, the order is the opposite to that of s_i and t_i.

Defining the fullshift $\Sigma_2 = \{\sigma = \sigma_0\sigma_1\sigma_2.... \text{ where } \sigma_i = L \text{ or } R\}$ to be the set of all possible infinite symbolic strings of L's and R's, then any given infinite symbolic sequence is a singleton in the fullshift space. The Bernoulli shift map $s : \Sigma_2 \to \Sigma_2$ is defined by

$$s(\sigma) = s(\sigma_0\sigma_1\sigma_2...) = \sigma_1\sigma_2\sigma_3...$$

In general, not all symbolic sequences correspond to the trajectory of an initial condition θ_0. Restricting the shift map to a subset of Σ_2 consisting of all the itineraries that are realizable yields the subshift $\Sigma \subset \Sigma_2$.

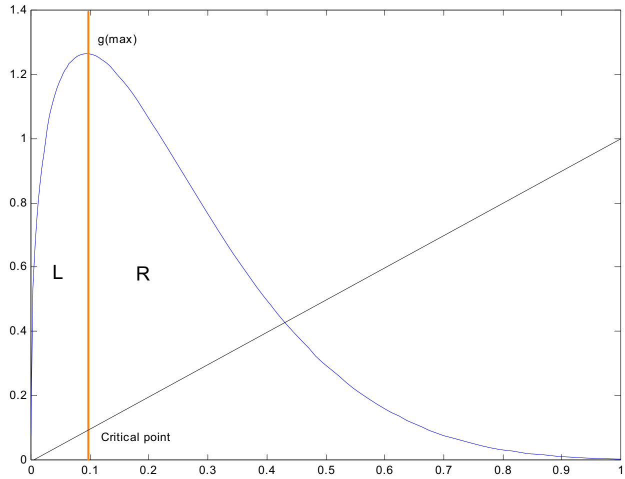

Figure 6: *Partition for the unimodal map g.*

If the initial condition is chosen to be the critical point, then the corresponding symbolic sequence (kneading sequence)[9] determines the topological entropy of the resulting subshift. We formulate the result in terms of topological Markov chains, a special class of subshifts of finite type where the transition in the symbol sequence is specified by a $0-1$ matrix. Any $(n \times n)$ binary matrix $M = (M_{ij})_{i,j=0,\ldots,n-1}$, $M_{ij} \in \{0,1\}$ generates a special subshift

$$\Sigma_M = \left\{\sigma \in \Sigma_2 : M_{\sigma_i \sigma_{i+1}} = 1, \ \forall i \in \mathbb{N}\right\}$$

which is called the topological Markov chain associated with M, and M is called the topological Markov matrix. We say that $M_{\sigma_i \sigma_{i+1}} = 1$ if the transition from σ_i to σ_{i+1} is possible. The matrix M gives a complete description of the dynamics of the unimodal map.

The premier numerical invariant of a dynamical system is its topological entropy h_{top}. For a subshift Σ

$$h_{top}(\Sigma) = \lim_{n\to\infty} \frac{\log\left(\sharp W_n(\Sigma)\right)}{n}$$

where $W_n(\Sigma)$ is the set of words of length n occurring in sequences of Σ. That is, the entropy is the exponential growth rate of the Σ-words. For a shift of finite type defined by a topological Markov matrix M, the topological entropy can be computed directly as the natural logarithm of the spectral radius of the generating transition matrix.

For the parameter values $a = 0.75$, $b = 0.58$, $d = 0.62$, $\alpha = 0.15$, we found a period 5 orbit: $\{1.8549, 0.0013, 0.4756, 1.1047, 0.1350\}$ which is shown in Figure 7 with the corresponding Markov partition. The critical point assumes the value

[9] See [27] and for a functorial approach to kneading theory see [1].

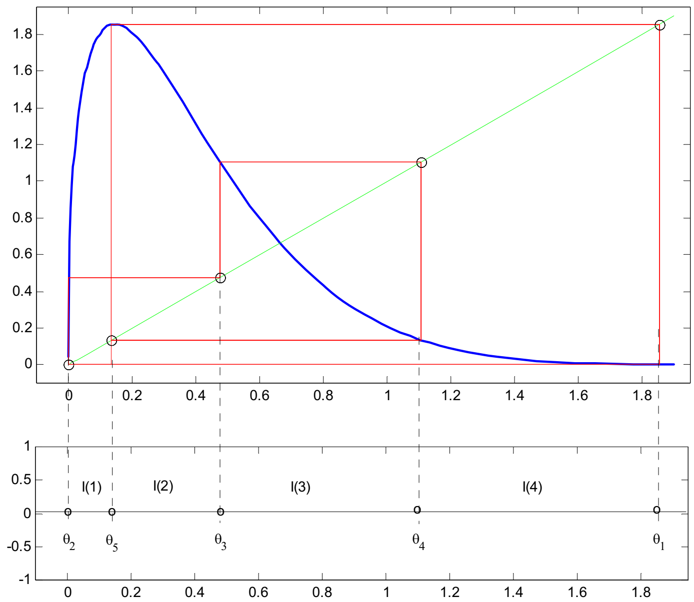

Figure 7: *Markov partition for a period 5 orbit.*

$\theta_{\max} = 0.1452$ and generates the symbolic partition for the map g. The periodic orbit has the following symbolic address:

$$(\theta_1\theta_2\theta_3\theta_4\theta_5)^\infty = (RLRRC)^\infty$$

and in consequence we have the following Markov matrix:

$$M_{RLRRC} = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{array} \begin{array}{c} \begin{array}{cccc} I_1 & I_2 & I_3 & I_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} .$$

The spectrum radius of this matrix is given by $\lambda = 1.5128$, which implies that the topological entropy is positive: $h_{top} \simeq 0.4140$ and this shows very clearly that we are dealing with chaotic motion in this set of parameter values. It should be noted that this is a very simple and rigorous way to estimate the topological entropy of a one-dimensional model and to check for the existence of chaos,

4 Control of Chaos

In this paper we also intend to show that chaotic dynamics may have a significant importance for economics not only on the modelling side, but also on the normative side by giving a possible new dimension to economic policy. To clarify this point we will apply the OGY method [31] to control the chaotic motion that is produced in this model.[10] For such purpose let us note the map by $g(\theta_t, b_t)$ in order to control the unstable period one orbit by applying a tiny perturbation to b, which is the parameter that we assume the government can affect through A. The control strategy is the following: find a stabilizing local feedback control law which is defined on a neighborhood of the desired periodic orbit. This is done by considering the first order approximation of the system at the chosen unstable periodic orbit. The ergodic nature of the chaotic dynamics of the model ensures that the state trajectory eventually enters into the neighborhood. Once inside the neighborhood, we apply the stabilizing feedback control law in order to steer the trajectory towards the desired orbit.

For values of θ_t close to the unstable fixed point θ_* and for values of b_t close to b_*, the map g can be approximated by the following linear discrete time system

$$x_{t+1} = Bx_t + Cy_t, \tag{5}$$

where $x_t = \theta_t - \theta_*$ and $y_t = b_t - b_*$ are the derivations from the nominal values in standard control notation for states and input. B and C represent the derivatives of the map g with respect to the variable and to the control parameter evaluated at the point (θ_*, b_*), that is

$$B = \left(\frac{\partial g}{\partial \theta}\right)_{(\theta_*, b_*)} \quad \text{and} \quad C = \left(\frac{\partial g}{\partial b}\right)_{(\theta_*, b_*)}.$$

Now according to OGY a linear state feedback

$$y_t = -Kx_t$$

is applied to system (5). It should be added that this control is only applied within a certain region

$$R_\varepsilon = \{\theta : |\theta - \theta_*| < \varepsilon\}, \varepsilon > 0$$

around the fixed point, which we will call the control region. Then, the system (5) will take the form

$$x_{t+1} = (B - CK)\, x_t,$$

and thus the closed loop system is stable as long as

$$|(B - CK)| < 1.$$

Setting $(B - CK) = 0$, then we have the pole placement technique and obviously $K = B/C$.

It was shown in the previous section that for $a = 0.9618$, $\alpha = 0.1556$, $d = 0.4585$, $b = 0.8$ the map g possesses an unstable chaotic fixed point $\theta_* = 0.5298$ (Example 1). We fix these parameter values and consider that b is the control parameter which is available for external adjustment but restricted to lie in some small interval $|b - b_*| < \varepsilon, \varepsilon > 0$ around the nominal value $b_* = 0.8$. Since

$$B = \left(\frac{\partial g}{\partial \theta}\right)_{(\theta_*, b_*)} = -2.0454 \text{ and } C = \left(\frac{\partial g}{\partial b}\right)_{(\theta_*, b_*)} = -1.9917,$$

[10] For controlling economic chaos in a model that produces *hyperchaos* see [26].

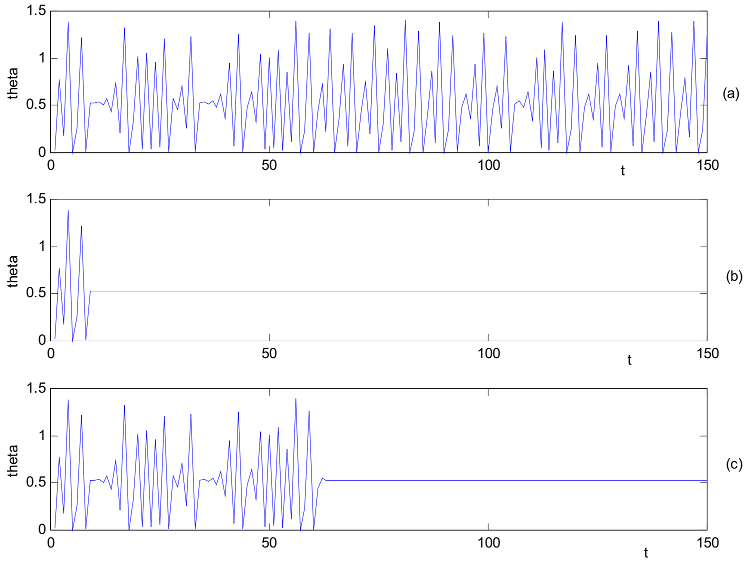

Figure 8: *(a) Chaotic trajectory; (b) Control switched on at $t = 1$; (c) Control switched on at $t = 50$.*

one obtains that

$$x_{t+1} = (-2.0454 + 1.9917K)\, x_t,$$

and by choosing the pole placement value for this last equation, it follows that $K = 1.02692$. For this value of the control matrix K the unstable period one orbit is stabilized, what can be seen in Figure 8, panels (b) and (c). Panel (a) shows the randomly chosen trajectory which we wished to steer towards the fixed point. The time to achieve control is very short, despite the initial condition or the moment when the control was switched on.

5 Concluding Remarks

In order to obtain relevant answers to whether there are or not chaotic dynamics under certain ranges of parameters values in a 1-dimensional particular model, we suggest that a bifurcation diagram, the variation of the Lyapunov exponent, the existence of a periodic point of period not equal to a power of two, and symbolic dynamics are very powerful techniques for that purpose. The application of these techniques in this paper clearly confirmed that a very simple model of a matching labor market, with well behaved aggregate functions (continuous, twice differentiable and linearly homogeneous) do really produce chaotic behavior for a large range of parameter sets, some of which had been questioned in [2].

Moreover, the irregular dynamics were easily controlled with a very small perturbation to one of models's parameters, the topological characteristics of the system remain the same, and the only aspect of the model that was changed was the type of stability of its fixed point: prior to the application of the control procedure, the fixed point was unstable, becoming and remaining stable as long as the control is left activated. If we think of this model as representing the dynamics of a true economy, contrary to a view that has been presented in the past, we do not have to change the nature of the system in order to control its chaotic dynamics: all we have to do is to impose small perturbations applied at the right time and on the right places. Why should chaos, as suggested in most papers – [2] included – be treated more as a curse for economic policy rather than a blessing? In most cases, it turns out to be much easier to control the dynamics of a chaotic system than other forms of dynamics.

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