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Stability under Learning: the Neo-Classical Growth Problem

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Abstract

A local stability condition for the standard neo-classical Ramsey growth model is derived. The proposed setting is deterministic, defined in discrete time and expectations are formed through adaptive learning.

Keywords: Neo-classical growth, Adaptive learning, Stability.

JEL classification: O41, C62, D83

1 Introduction

Modern macroeconomics have progressively replaced the notion that agents are fully rational by a concept of learning, under which individuals and firms collect information over time, learn with this information and eventually accomplish a long term capacity to produce optimal decisions. Rational expectations / perfect foresight emerge, then, more reasonably as a long run possibility and not as an every period underlying assumption. The literature on macroeconomic learning is extensive and covers almost all the relevant phenomena, like monetary policy, asset pricing or growth, and the analysis is undertaken both in stochastic and deterministic environments.

Our concern is with deterministic economic growth and the aim is to derive a simple and straightforward condition of stability for the neo-classical Ramsey growth model. Deterministic models of growth under learning have been analyzed before in the literature, but essentially in the context of overlapping generations models, e.g. by Bullard (1994) Sorger (1998) and Schonhofer (1999). A relevant exception is Cellarier (2006) who effectively considers the intertemporal utility maximization setup; however, the concern of this author is essentially with the search for endogenous business cycles in a scenario where convergence to rational expectations is excluded from the start.

The relevant parameter in the analysis to undertake is the steady state level of the gain sequence. This parameter indicates whether the learning process was successful (in the sense that it allows for asymptotic perfect foresight) or not. Optimal learning requires the gain sequence to converge to zero; otherwise, if it converges to any value between 0 and 1 the learning process is not efficient (the more it departs from 0, the larger is the degree of inefficiency). The absence of a perfect process of learning must not be interpreted as an uncommon or even an undesired outcome; taking the words of Sobel (2000), 'Agents in these models begin with a limited understanding of the problems that they must solve. Experience improves their decisions. Death and a changing environment worsen them.' (page 241), and, furthermore, 'An agent will not necessarily learn the optimal decision when the cost of acquiring additional information exceeds the benefits.' (page 244).

A relevant feature of many learning mechanisms, as the adaptive learning setup we consider, is that agents do not necessarily need to accomplish the rational expectations long term outcome to generate exactly the same steady state result as if they did. Rather, there is generally a minimal requirement in terms of the long term capacity of predicting future values that produces precisely the same result as under perfect foresight. If learning is costly (and, effectively, there are always costs in acquiring and processing information), then the effort on reaching an optimal forecasting capacity does not pay; the agent benefits in locating at the point in which: (*i*) stability at the perfect foresight level of the considered endogenous variables holds; (*ii*) the costs of learning are the lowest possible. Below, we derive a straightforward condition for stability that reveals that the higher is the level of technology and the lower are the discount rate of future utility and the depreciation rate of capital, the less the representative agent will need to learn in order to accomplish the intended long term result.

The remainder of this note is organized as follows. Section 2 presents the structure of the growth model and introduces the adaptive learning mechanism. Section 3 explains how to transform the model into a linearized system in the neighborhood of the steady state, allowing for a local stability analysis. The stability condition is derived in section 4. Section 5 concludes.

2 The Growth Model and the Learning Mechanism

Consider a standard one-sector optimal growth model. A representative agent maximizes consumption utility intertemporally, under an infinite horizon and taking a positive future utility discount rate, ρ. Thus, the agent maximizes $V_0 = \sum_{t=0}^{+\infty} \left(\frac{1}{1+\rho}\right)^t U(c_t)$, with $U(c_t) : \mathbb{R}_+ \to \mathbb{R}$ the instantaneous utility function; variable c_t represents per capita consumption. The utility function must obey to trivial conditions of continuity and differentiability, and marginal returns must be positive and diminishing. To aid on the tractability of the model, we assume a simple logarithmic utility function $U(c_t) = \ln c_t$.

The resource constraint of the problem is the conventional capital accumulation equation: $k_{t+1} = f(k_t) - c_t + (1-\delta)k_t$, k_0 given. Variable $k_t \geq 0$ represents the per capita stock of capital and $\delta > 0$ refers to the rate of capital depreciation. The production function is neo-classical, i.e., it evidences decreasing marginal returns. Assuming a Cobb-Douglas production technology, we consider $f(k_t) = Ak_t^\alpha$, with $A > 0$ the technology index and $\alpha \in (0,1)$ the output-capital elasticity.

Maximizing V_0 subject to the resource constraint, one derives three first-order conditions: $E_t p_{t+1} = 1/c_t$; $[1 + \alpha A k_t^{-(1-\alpha)} - \delta] E_t p_{t+1} = (1+\rho)p_t$; $\lim_{t\to+\infty} k_t \left(\frac{1}{1+\rho}\right)^t p_t = 0$ (transversality condition). In these conditions, p_t stands for the shadow-price of capital and $E_t p_{t+1}$ is the expected value of the shadow-price for the subsequent time period. From the first optimality condition, we infer that $E_{t+1} p_{t+2} = 1/E_t c_{t+1}$, and therefore we resort to the second optimality condition to present an equation of motion for the next period expected per capita consumption level,

$$E_t c_{t+1} = \frac{1 + \alpha A k_{t+1}^{-(1-\alpha)} - \delta}{1+\rho} c_t \tag{1}$$

The perfect foresight steady state for the system composed by the capital constraint and equation (1) is obtained by imposing $\overline{k} := k_{t+1} = k_t$ and $\overline{c} := E_t c_{t+1} = c_{t+1} = c_t$. Straightforward computation conducts to the unique steady state pair of values $(\overline{k}, \overline{c}) = \left[\left(\frac{\alpha A}{\rho+\delta}\right)^{1/(1-\alpha)}; \frac{1}{\alpha}(\rho + (1-\alpha)\delta)\overline{k}\right]$. Under perfect foresight, the system is saddle-path stable, i.e., if the one-dimensional stable path is followed, the convergence towards the steady state point is fulfilled.

Assume that expectations about the next period level of consumption are formed through adaptive learning. Following Adam, Marcet and Nicolini (2008), we consider an estimator variable b_t such that $E_t c_{t+1} = b_t c_t$. The estimator is updated taking into account past information and using the rule $b_t = b_{t-1} + \sigma_t \left(\frac{c_{t-1}}{c_{t-2}} - b_{t-1}\right)$, b_0 given. Variable $\sigma_t \in (0,1)$ respects to the gain sequence, as characterized in the introduction. We do not need to explicitly model the time evolution of this variable because we will concentrate the analysis in the long run properties of the growth system. It is simply necessary to know that if σ_t converges to zero ($\overline{\sigma} = 0$), a steady state perfect foresight result is attained (i.e., the unique steady state point is accomplished), while if σ_t converges to any positive value lower than 1, then a less than optimal long run forecasting ability is evidenced (the higher is $\overline{\sigma}$, the lower will be the steady state quality of the forecasts).

3 Linear Approximation in the Neighborhood of the Steady State

The goal is to analyze local stability conditions, i.e., conditions under which convergence to $(\overline{k}, \overline{c})$ is accomplished, for a given pair (k_0, c_0) close to equilibrium. Working in the neighborhood of the steady state point, we linearize the system of difference equations relating to the motion of capital and consumption in order to attain stability conditions.

The linearization procedure is undertaken in two steps. First, we linearize function $F(k_t, c_t) := E_t c_{t+1}/c_t$; this allows to write the estimator as a linear function of the two endogenous variables, opening the way for explicitly presenting a system of capital-consumption equations defined in terms of contemporaneous and past values of variables. Second, we linearize the obtained system in order to build a Jacobian matrix from which stability conditions are straightforward to derive.

Given the relation between expected consumption, present consumption and the estimator, in the neighborhood of the steady state we can write: $b_t \simeq 1 + F_k(\overline{k}, \overline{c})(k_t - \overline{k}) + F_c(\overline{k}, \overline{c})(c_t - \overline{c})$. Straightforward computation allows to find $F_k(\overline{k}, \overline{c}) = -(1-\alpha)(\rho+\delta)/\overline{k}$ and $F_c(\overline{k}, \overline{c}) = (1-\alpha)(\rho+\delta)/((1+\rho)\overline{k})$. Therefore, defining $\theta := 1 + \frac{\alpha-(1-\alpha)(\rho+\delta)}{\alpha(1+\rho)}(1-\alpha)(\rho+\delta)$, one arrives to

$b_t \approx \theta - (1-\alpha)(\rho+\delta)\frac{k_t}{\overline{k}} + (1-\alpha)\frac{(\rho+\delta)}{(1+\rho)}\frac{c_t}{\overline{k}}$. Replacing this expression in the updating estimator rule, the following difference equation for consumption is obtained,

$$c_t \simeq (1-\sigma_t)[c_{t-1} - (1+\rho)k_{t-1}] + \sigma_t(c_{t-1}/z_{t-1} - \theta) + (1+\rho)k_t;\ z_t = c_{t-1} \quad (2)$$

The second step of the linearization procedure consists in taking the capital equation and the pair of equations (2) and evaluating them in the neighborhood of the steady state. A three dimensional matricial system emerges,

$$\begin{bmatrix} k_t - \overline{k} \\ c_t - \overline{c} \\ z_t - \overline{c} \end{bmatrix} \simeq$$

$$\begin{bmatrix} 1+\rho & -1 & 0 \\ (1+\rho)(\rho+\overline{\sigma}) & (1-\overline{\sigma}) + \overline{\sigma}/\overline{c} - (1+\rho) & -\overline{\sigma}/\overline{c} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_{t-1} - \overline{k} \\ c_{t-1} - \overline{c} \\ z_{t-1} - \overline{c} \end{bmatrix} \quad (3)$$

4 Stability Condition

Let J be the 3×3 Jacobian matrix in (3). For this matrix, it is straightforward to compute determinant, sum of principle minors and trace. They are all positive values: $Det(J) = (1+\rho)\frac{\overline{\sigma}}{\overline{c}} > 0$; $\Sigma M(J) = (2+\rho)\frac{\overline{\sigma}}{\overline{c}} > 0$; $Tr(J) = (1-\overline{\sigma}) + \frac{\overline{\sigma}}{\overline{c}} > 0$.

Stability conditions involving determinant, sum of principle minors and trace of a three-dimensional linearized system are the following [see Brooks (2004)]:

(i) $1 - Det(J) > 0$;

(ii) $1 - \Sigma M(J) + Tr(J)Det(J) - (Det(J))^2 > 0$;

(iii) $1 - Tr(J) + \Sigma M(J) - Det(J) > 0$;

(iv) $1 + Tr(J) + \Sigma M(J) + Det(J) > 0$.

In the specific case under analysis, we observe that $\Sigma M(J) = Tr(J) + Det(J) - (1-\overline{\sigma})$. Thus, the stability conditions are reduced to:

(i) $1 - Det(J) > 0$;

(ii) $2 - \overline{\sigma} - Tr(J) - Det(J) + Tr(J)Det(J) - (Det(J))^2 > 0$;

(iii) $\overline{\sigma} > 0$;

(iv) $\overline{\sigma} + 2Tr(J) + 2Det(J) > 0$.

Conditions (iii) and (iv) are verified for any values of parameters obeying the imposed constraints. Condition (i) requires $\overline{\sigma} < \overline{c}/(1+\rho)$ and condition (ii) implies $-\frac{1+\sqrt{1+4(1+\rho)(\overline{c}+\rho)}}{2(\overline{c}+\rho)}\frac{\overline{c}}{1+\rho} < \overline{\sigma} < \frac{\sqrt{1+4(1+\rho)(\overline{c}+\rho)}-1}{2(\overline{c}+\rho)}\frac{\overline{c}}{1+\rho}$. Let $\phi := \frac{\sqrt{1+4(1+\rho)(\overline{c}+\rho)}-1}{2(\overline{c}+\rho)}$. Condition (i) will be more restrictive than condition (ii) if $\phi > 1$. This last inequality would imply $\overline{c} < 0$, which is not a feasible outcome. Therefore, the first condition can be set aside and, hence, the unique relevant stability condition is the upper bound of (*ii*) (note that the lower bound is below zero, and consequently it can be ignored). This result is presented in the form of a proposition,

Proposition 1 *In the neo-classical Ramsey growth model with expectations generated through adaptive learning, stability holds under condition* $\overline{\sigma} < \phi\overline{c}/(1+\rho)$*, with* $0 < \phi < 1$*.*

The result in proposition 1 is intuitive. It sets a boundary on learning inefficiency or, in other words, it presents a minimum requirement in terms of information acquisition and processing needed in order for the steady state to be accomplished. As discussed in the introduction, assuming a costly learning process, the representative agent does not need to employ resources to attain $\overline{\sigma} = 0$. She just has to apply a level of effort that is enough to guarantee that $\overline{\sigma}$ is close to, but below, $\phi\overline{c}/(1+\rho)$.

Proposition 2 briefly states the determinants of the learning boundary.

Proposition 2 *The learning requirements are relaxed (i.e., the representative agent has to make less learning effort in order to reach the steady state result) with a relatively higher level of technology and with lower depreciation and discount rates.*

The results in proposition 2 follow directly from observing that $\partial\overline{c}/\partial A > 0$, $\partial\overline{c}/\partial\delta < 0$ and $\partial\overline{c}/\partial\rho < 0$ (and noticing that $\partial\phi/\partial\overline{c} > 0$).

To close the analysis, a numerical example is presented. The benchmark values of parameters are $\alpha = 0.3$, $\delta = 0.05$ (per year), $\rho = 0.02$ (per year).[1] Parameter A is chosen to guarantee $\overline{k} = 1$, i.e., $A = 0.233$. In this case, $\overline{c} = 0.183$ and the stability condition is $\overline{\sigma} < 0.156$. The gain sequence must possess a steady state value lower than 0.156 in order to allow for stability / convergence to the steady state pair $(\overline{k}, \overline{c}) = (1, 0.183)$.

Results in proposition 2 can be illustrated by varying some of the parameter values. In table 1, various experiments are displayed.

Parameter values*	$\overline{k}$	$\overline{c}$	Stability condition
$\delta = 0.02$	2.220	0.252	$\overline{\sigma} < 0.205$
$\delta = 0.1$	0.462	0.139	$\overline{\sigma} < 0.122$
$\rho = 0.01$	1.244	0.187	$\overline{\sigma} < 0.160$
$\rho = 0.05$	0.600	0.170	$\overline{\sigma} < 0.142$
$A = 0.2$	0.802	0.091	$\overline{\sigma} < 0.082$
$A = 0.5$	2.971	0.545	$\overline{\sigma} < 0.387$

Table 1 - Stability condition for different values of parameters (*The other parameters maintain the proposed benchmark values).

[1] These values are withdrawn from Barro and Sala-i-Martin (1995), pages 78-79.

The stability conditions in the table confirm the results in proposition 2: to attain stability, learning becomes more demanding when the depreciation rate of capital is higher, the discount rate of future utility is higher and the level of technology regresses.

5 Final Remarks

This note has derived an explicit, simple and intuitive stability condition for the conventional Ramsey growth model when expectations about future consumption are formed through adaptive learning. The relevance of the result is that the representative agent may be boundedly rational (i.e., she may not be able to treat information with the efficiency needed in order to achieve a long run optimal forecasting capability), and still be able to attain the intended long run locus (the unique steady state point). Nevertheless, there is a clear boundary: after some threshold value of learning inefficiency, stability is lost. A high technological capacity and low capital depreciation and intertemporal discount rates allow to relax the learning constraint.

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