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Abstract

The main aim of this paper is to analyse the dynamics of nonlinear discrete-time maps generated by duopoly games with heterogeneous and quadratic cost functions, in which players do not form expetations about the rival's actions accordingto the ratioal expectations hypothesis. We discusse here two cases. In the first one we introduce games with boundedly rational players and in the second one games with adaptive expectations. The dynamics are mainly analysed by numerical simulations. There are always multiple equilibria, and the significance of the Nash equilibria is pointed out.

1 Introduction

The complexity of oligopoly models may arise from a large set of sources. It may come from the particular form of expectations formation related to the rival actions that is considered in the model, or from the kind of cost structures that are taken into consideration, or, finally, whether the demand function is linear or not. In this paper we assume that the market demand function is linear and that cost structures are nonlinear and heterogeneous. What happens to the dynamics of the Cournot model if the firms form expectations not according to the rational expectations hypothesis?

What happens if both firms are bounded rational? By bounded rational firms, we have in mind that firms usually do not have a complete knowledge of the market, and hence they try to use partial information based on the local estimates of the marginal profit. At each time period t, each firm increases (decreases) its production q_i at the period $(t+1)$ if the marginal profit is positive (negative). This kind of expectations seem to be more realistic than the rational expectations for two reasons. Firstly, because it is

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not assumed that the players have complete information of all the underlying features of the market, including the actions of their rivals period by period, and secondly, firms adjust their outputs based on local estimates of marginal profits allowing the game to played in a framework where the process towards the Nash equilibria can be learned or not.

Do the dynamics of the model suffer a significant change if we introduce a large stickiness into its structure. For example, one may question whether the large instability and the multiplicity of equilibria that arises from the previous scenario is somewhat changed if the one of the two oligopolists form expectations according to naive expectations. Naive expectations appear in the early work of Cournot and means that in every step, each player expects his rival to offer the same quantity for sale in the current period as it did in the preceding one. He showed that this process converge to the unique intersection point of the two reaction functions, which actually is the fixed point of the system and is known as the Cournot-Nash equilibrium [6].

As we will show, a very simple economic structure may lead to extremely complex dynamics, to multiple equilibria and to chaotic dynamics in both scenarios. A major result obtained is that the Nash equilibrium (the noncooperative solution of the game) changes from stable and periodic to chaotic, through period-doubling bifurcations, when the system parameters are changed. The existence of chaotic motion is exposed by numerical and graphical analysis.

The paper is organized as follows. Section two deals with the analysis of expectations. Section three presents the study of the dynamics with homogeneous and with heterogeneous expectations, while the final section concludes.

2 Expectations

We consider a dynamic version of a simple Cournot-type duopoly market where players produce homogeneous goods which are perfect substitutes and offer them at discrete-time periods $t = 0, 1, 2, ...$ in a common market. At each period t, every firm must form an expectation of the rival's output in the next time period in order to determine the corresponding profit-maximizing quantities for period $t + 1$.

If we denote by $q_i(t)$, $i = 1, 2$ the output of firm i at time t, then its production $q_i(t+1)$, $i = 1, 2$ for the next period $t + 1$ is decided by solving the two optimization problems

$$\begin{cases} q_1(t+1) = \arg\max_{q_1} \ \Pi_1(q_1(t), q_2^e(t+1)) \\ q_2(t+1) = \arg\max_{q_2} \ \Pi_2(q_1^e(t+1), q_2(t)), \end{cases} \tag{1}$$

where the function $\Pi(\cdot,\cdot)$ denotes the profit of the i^{th} firm and $q_j^e(t+1)$ represents the expectations of firm i about the production decision of firm

j $(j = 1, 2, j \neq i)$. If the optimization problems have unique solutions, then we denote them by

$$\begin{cases} q_1(t+1) = f(q_2^e(t+1)) \\ q_2(t+1) = g(q_1^e(t+1)), \end{cases}$$

where f and g are called the reaction functions.

The most simple case is when both players choose the strategy defined by naive expectations, that is $q_i^e(t+1) = q_i(t)$, for $i = 1, 2$, and then the model becomes

$$\begin{cases} q_1(t+1) = f(q_2(t)) \\ q_2(t+1) = g(q_1(t)), \end{cases}$$

which is known as a anti-triangular map, whose dynamics can be studied by using analytical and numerical tools [6].

Since information in the market is far from complete, the players can use less complicated expectations formation processes such as bounded rationality methods. The bounded rational firms do not have a complete knowledge of the market, hence they make their output decisions based on a local estimate of the marginal profit $\partial\Pi_i/\partial q_i$. A firm decides to increase its production q_i if it has a positive marginal profit, or decrease its production if the marginal profit is negative. We denote the bounded rational players by 1 and 2. Then the dynamical equations of players 1 and 2 have the form:

$$\begin{cases} q_1(t+1) = q_1(t) + \phi_1 q_1(t) \dfrac{\partial\Pi_1}{\partial q_1(t)} \\ q_2(t+1) = q_2(t) + \phi_2 q_2(t) \dfrac{\partial\Pi_2}{\partial q_2(t)} \end{cases} \tag{2}$$

where ϕ_1, ϕ_2 are positive parameters which represent the relative speed of adjustment of each player.

Another expectation rule that firms can use to revise their beliefs is according to the adaptive expectation rules. If firms 1 and 2 react according to adaptive expectations, then they compute its output with weights between last period's outputs $q_1(t)$ and $q_2(t)$ and its reaction function $f(q_2)$ and $g(q_1)$. Hence the dynamic equations of the adaptive expectation players have the form

$$\begin{cases} q_1(t+1) = (1-v_1)\, q_1(t) + v_1 f(q_2(t)) \\ q_2(t+1) = (1-v_2)\, q_2(t) + v_2 g(q_1(t)) \end{cases}$$

where $0 < v_1, v_2 < 1$ represent the speed of adjustment of each adaptive player.

All these nonlinear discrete-time duopoly games are homogeneous, and to consider heterogeneous cases we have to choose players with different expectations or different cost curves.

3 The model with heterogeneous cost structures

We consider a duopoly model where the inverse demand function is assumed linear and decreasing:

$$P = p(Q) = a - b(q_1 + q_2), \tag{3}$$

where $Q = q_1 + q_2$ is the industry output and $a, b > 0$. Following Kopel's approximation, the market is supplied by two firms with nonlinear cost functions

$$C_1(q_1, q_2) = c_1(q_2)\, q_1 \text{ and } C_2(q_1, q_2) = c_2(q_1)\, q_2, \tag{4}$$

that is, each firm has a marginal cost of production that is constant with respect to its own output but varies with respect to the rival's output.

We consider the following specific form (see Nonaka, 2003) for the functions $c_i,\ i = 1, 2:$

$$\begin{cases} c_1(q_2) = a - bq_2 - 2b(\alpha q_2 - \alpha + 1)^2 \\ c_2(q_1) = a - bq_1 - 2b(\beta q_1 - 1)^2. \end{cases}$$

This last relation implies that duopoly firms have unequal nonlinear marginal cost of production and therefore production externalities are nonlinear for both firms and heterogeneous between the firms.

According with (3),(4),(1) we have now the following profit functions for the firms:

$$\begin{cases} \Pi_1 = pq_1 - c_1(q_2)\, q_1 \\ \Pi_2 = pq_2 - c_2(q_1)\, q_2 \end{cases}$$

and the reaction functions are given by $f(q_2) = (\alpha q_2 - \alpha + 1)^2$ and $g(q_1) = (\beta q_1 - 1)^2$.

3.1 Bounded rational players

In this case we consider a game with two bounded rational players which is given by the nonlinear discrete-time map defined in (2) which is equivalent to

$$\begin{cases} q_1(t+1) = q_1(t) + \phi_1 q_1(t)\left(-2bq_1(t) + 2b(\alpha q_2(t) - \alpha + 1)^2\right) \\ q_2(t+1) = q_2(t) + \phi_1 q_2(t)\left(-2bq_2(t) + 2b(\beta q_1(t) - 1)^2\right) \end{cases},$$

and where $\phi_1, \phi_2 > 0$ are the respective speeds of adjustment. The map depends on 5 parameters, but for simplicity, we only focus on the parameters that appear in both equations of the system, that is $b > 0$ and $\phi_1, \phi_2 > 0$.

We can find the equilibrium points by solving the following system of nonlinear equations

$$\begin{cases} q_1(t) = q_1(t) + \phi q_1(t)\left(-2bq_1(t) + 2b(\alpha q_2(t) - \alpha + 1)^2\right) \\ q_2(t) = q_2(t) + \phi q_2(t)\left(-2bq_2(t) + 2b(\beta q_1(t) - 1)^2\right) \end{cases} \tag{5}$$

Proposition 1 *The solutions of the system (5), that is* $(q_1, q_2) = (0, 0)$, $(q_1, q_2) = (0, 1)$ *and* $(q_1, q_2) = \left((1-\alpha)^2, 0\right)$, *are fixed points for the game with two bounded rational players for any setting of the parameters values. These fixed points are generally unstable and with no significance in terms of Nash equilibria. The other possible equilibrium points are given by the solution(s) of the nonlinear system*

$$\begin{cases} q_1(t) = (\alpha q_2(t) - \alpha + 1)^2 \\ q_2(t) = (\beta q_1(t) - 1)^2 \end{cases},$$

which is the model for the case of naive expectations. There are always 2, 3 or 4 real solution which are given by the intersection points of the parabolas represented in Figure 1, and the stability of these points vary with the parameters setting.

Proof. System (5) is equivalent to

$$\begin{cases} q_1(t)\left(-q_1(t) + (\alpha q_2(t) - \alpha + 1)^2\right) = 0 \\ q_2(t)\left(-q_2(t) + (\beta q_1(t) - 1)^2\right) = 0 \end{cases}, \tag{6}$$

and to solve this we have to form 4 subsystems which gives the solutions refereed above. In order to study the stability of the equilibrium point we have to consider the Jacobian matrix, that is

$$J = \begin{bmatrix} 1 - 4bq_1\phi + 2b\phi(\alpha q_2 - \alpha + 1)^2 & 4\alpha b\phi q_1(\alpha q_2 - \alpha + 1) \\ 4b\phi\beta q_2(\beta q_1 - 1) & 1 - 4bq_2\phi + 2b\phi(\beta q_1 - 1)^2 \end{bmatrix}.$$

We conclude that the fixed point $(q_1, q_2) = (0, 0)$ is always unstable since the Jacobian matrix takes the following diagonal form

$$J(0,0) = \begin{bmatrix} 1 + 2b\phi(1-\alpha)^2 & 0 \\ 0 & 1 + 2b\phi \end{bmatrix}$$

and since the eigenvalues are given by the entries of the principal diagonal, that is

$$\begin{aligned} \lambda_1 &= 1 + 2b\phi(1-\alpha)^2 > 1, \ \forall b > 0, \phi > 0, 0 < \alpha < 1 \\ \lambda_2 &= 1 + 2b\phi > 1, \ \forall b > 0, \phi > 0. \end{aligned}$$

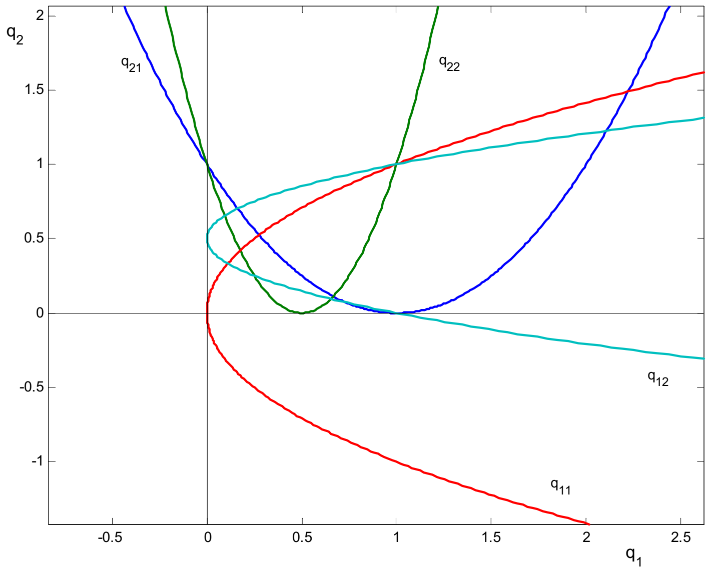

Figure 1: Fixed points for the game with two boundedly rational players

Similarly, we obtain that the fixed point $(q_1, q_2) = (0, 1)$ is a saddle point if $b\phi > 1$ and is unstable (both eigenvalues greater than one) in other cases. The equilibrium point defined by $(q_1, q_2) = \left((1-\alpha)^2, 0\right)$ is a saddle point if $\phi b (1-\alpha)^2 > 1$ and is unstable in the other cases.

In Figure 1 we represent the two quadratic maps from system (6) for $\alpha = 1, 2$ and $\beta = 1, 2$. We denote this parabolas by q_{ij}, where $i = 1, 2$ define the variable and $j = 1, 2$ define the parameters values. We can observe that for $\alpha = 1$ and $\beta = 1$ there are 2 real fixed points, namely $(0.2755, 0.5248)$ and $(2.2207, 1.4902)$, and for $\alpha = 2$ and $\beta = 2$ there are 4 intersections between the two parabolas, that is 4 fixed points, namely $(1, 1)$, $(1/4, 1/4)$, $(0.0954, 0.6545)$ and $(0.6545, 0.0954)$. The number of fixed points of the sub-system (6) depends only on the variation of the parameters α and β and it was studied by Nonaka (2003), who found some parameter regions where the sub-system has 2, 3 or 4 fixed points.

If we now consider $\alpha = 1.8$ and $\beta = 1.1$, the parameter values that we will use in the numerical simulations from the next section, we obtain the following two fixed points: $(q_1^*, q_2^*) = (0.1649, 0.6700)$ and $(q_1', q_2') = (1.9100, 1.2122)$. The Jacobian matrix computed at the first fixed point is

$$J(0.1649, 0.6700) = \begin{bmatrix} 1 - 0.3299\phi b & 0.4820\phi b \\ -2.4132\phi b & 1 - 1.3397\phi b \end{bmatrix}$$

and using the standard condition for stability of the considered Nash equilibrium we obtain

$$\left\{ \begin{array}{l} 1 + trJ + \det J = 4 - 3.3393\phi b + 1.6053\,(\phi b)^2 > 0 \\ 1 - trJ + \det J = 1.6053\,(\phi b)^2 > 0 \\ 1 - \det J = 1.6696\phi b - 1.6053\,(\phi b)^2 > 0 \end{array} \right. .$$

The first condition is always satisfied, since the discriminant is negative and the coefficient of $(\phi b)^2$ is positive, the second condition is always satisfied since it is a positive expression and the third condition is satisfied if $\phi b > 1.0401$. Moreover for $\phi b = 1.0401$ we have a Neimark-Saker bifurcation, where a pair of complex conjugate eigenvalues with modulus one are encountered. The complex behavior which is developed as a consequence of a first Neimark-Saker bifurcation is presented in the next section, in particular in Figure 2 and 3.

The second fixed point is always unstable since

$$\left\{ \begin{array}{l} 1 + trJ + \det J = 4 - 12.4895\phi b + 102.34\,(\phi b)^2 > 0 \\ 1 - trJ + \det J = -102.34\,(\phi b)^2 < 0 \\ 1 - \det J = 6.2447\phi b + 102.34\,(\phi b)^2 > 0 \end{array} \right.$$

that is, the second stability condition is not satisfied for any choice of the parameters ϕ and b.

It is interesting to observe that in this case, the only significant Nash equilibrium is the fixed point $(q_1^*, q_2^*) = (0.1649, 0.6700)$, all others equilibria being unstable. ■

3.1.1 Numerical simulations

The most important parameters are b and ϕ, which are those that mostly produces changes in the behavior of the duopoly model with two rationally bounded players. From the above proposition, we have that the only interesting Nash equilibrium is the fixed point $(q_1^*, q_2^*) = (0.1649, 0.6700)$, the others are always unstable. We study the behavior of this fixed point, when the parameters ϕ and b are varied.

We fix $\alpha = 1.8, \phi = \phi_1 = \phi_2 = 0.6, \beta = 1.1$ and let b vary between 1.5 and 2.362. The bifurcation diagram from Figure 2 illustrates the complex behavior of the nonlinear duopoly model, changing from stable equilibrium to chaotic trajectories, through Neimark-Saker and period-doubling bifurcations.

The first Neimark-Saker bifurcation occurs for $b = 1.7335$ and this parameter value is obtained from the solution of the following conditions:

$$\left\{ \begin{array}{l} 1 + trJ + \det J = 4 - 2.0036b + 0.5779b^2 > 0 \\ 1 - \det J = 1.0018b - 0.5779b^2 = 0 \end{array} \right. .$$

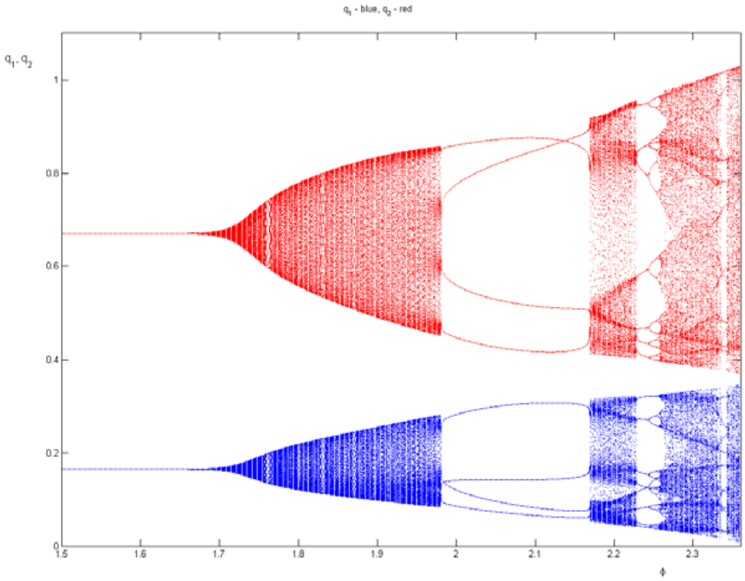

Figure 2: Bifurcation diagram when the parameter b varies between 1.5 and 2.36. (q_1 blue and q_2 red)

For this parameter setting the equilibrium point $(q_1^*, q_2^*) = (0.1649, 0.6700)$ is loosing stability since the complex conjugate eigenvalues $0.1315 \pm 0.9915i$ are on the border of the unit circle, since $|0.1315 \pm 0.9915i| = 1$. After the Neimark-Saker bifurcation a stable invariant closed curve is produced. If we further increase the value of parameter b, the closed invariant curve breaks and looses stability giving rise to a period four stability window. By the same way, other stability windows can be observed, some of them evolving in period doubling bifurcations and finally bifurcating in chaotic regions.

Figure 3 illustrates some different attractors of the duopoly model with boundedly rational players, for the parameter calibration presented above and when parameter b takes several different values. The first image shows the stable fixed point (q_1^*, q_2^*) , the second one present a period 12 orbit which appears in the last stability window of the bifurcation diagram ($b = 2.34$), the third image presents the break-down of the four closed invariant curves obtained after the Neimark-Saker bifurcation of the period four orbit showed in the predominant stability windows of the bifurcation diagram ($b = 2.2$), and finally, the fourth image illustrates the strange attractor of the system, where all orbits are settled down.

If we want to compute the topological entropy of this map, since it is a generic two-dimensional nonlinear map and since there are no analytical

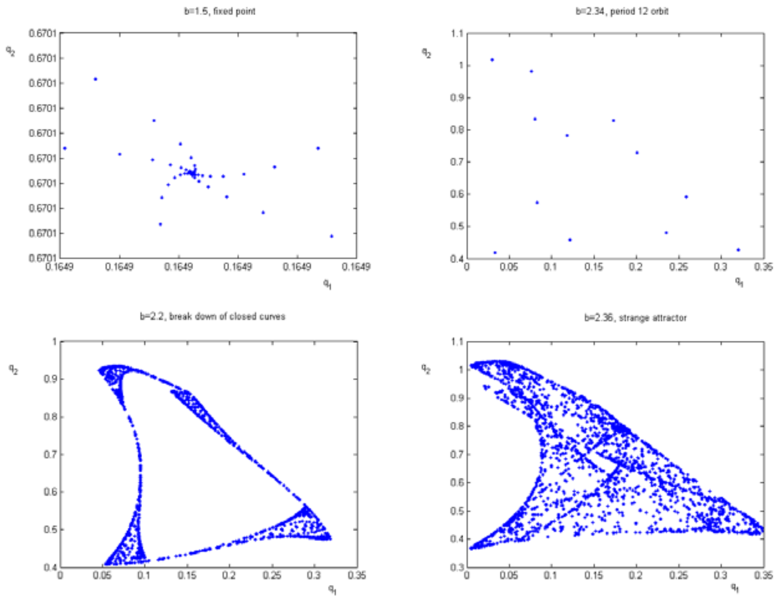

Figure 3: Several atractors when the parameter b is varied

or general methods to do this, we have to resort to numerical algorithms (which are also very few). We use the algorithm developed by Newhouse and Pignataro (1993) and we obtain that the topological entropy is estimated to be $h(T) = 0.4266...$ which means that we are dealing with chaotic motion, when the Nash equilibrium looses stability.

3.2 Introducing an adaptive player

If now we consider a bounded rational player and a player with adaptive expectations we obtain the following nonlinear model:

$$\begin{cases} q_1(t+1) = q_1(t) + \phi q_1(t) \dfrac{\partial \Pi_1}{\partial q_1(t)} \\ q_2(t+1) = (1-v) q_2(t) + v g(q_1(t)) \end{cases}$$

that is

$$\begin{cases} q_1(t+1) = q_1(t) + \phi q_1(t) \left(-2b q_1(t) + 2b (\alpha q_2(t) - \alpha + 1)^2 \right) \\ q_2(t+1) = (1-v) q_2(t) + v (\beta q_1(t) - 1)^2 \end{cases},$$

with $\phi > 0$, $0 < v < 1$, $b > 0$ and $1 < \alpha, \beta < 2$.

We find the equilibrium points by solving the following system of non-linear equation

$$\begin{cases} q_1(t) = q_1(t) + \phi q_1(t)\left(-2bq_1(t) + 2b(\alpha q_2(t) - \alpha + 1)^2\right) \\ q_2(t) = (1-v)\, q_2(t) + v(\beta q_1(t) - 1)^2 \end{cases}.$$

The point $(q_1, q_2) = (0, 1)$ is always a (boundary) fixed point and the others equilibrium points are given by the solution(s) of the system

$$\begin{cases} q_1(t) = (\alpha q_2(t) - \alpha + 1)^2 \\ q_2(t) = (\beta q_1(t) - 1)^2 \end{cases}.$$

There can be 0, 2,3 or 4 real solution for this system (the same as the case presented in the previous section) depending on certain conditions on the system parameters α and β.

The Jacobian matrix has the form

$$J = \begin{bmatrix} 1 - 4bq_1\phi + 2b\phi(\alpha q_2 - \alpha + 1)^2 & 4\alpha b\phi q_1(\alpha q_2 - \alpha + 1) \\ 2\beta v(\beta q_1 - 1) & 1 - v \end{bmatrix}$$

and the fixed point $(q_1, q_2) = (0, 1)$ is always a saddle point since

$$J(0,1) = \begin{bmatrix} 1 + 2b\phi & 0 \\ -2\beta v & 1 - v \end{bmatrix}$$

and the eigenvalues are given by the entries of the principal diagonal, that is

$$\begin{aligned} \lambda_1 &= 1 + 2b\phi > 1,\ \forall b > 0, \phi > 0 \\ \lambda_2 &= 1 - v < 1,\ \forall 0 < v < 1. \end{aligned}$$

The stability of the other fixed point, we should study by numerical simulations, since there is lack of analytical expression for these equilibria.

3.2.1 Numerical simulations

Now we consider that $\alpha = 1.5$, $\beta = 1.2, b = 1.5, \phi = 1.1$ and let the parameter v to vary between 0.55 and 0.6649. The bifurcation diagram of the variable q_1 is presented in Figure 4, where a period-doubling route to chaos can be observed.

For these setting of parameters there are three real fixed points, that is $(0, 1), (0.10967, 0.75411)$ and $(1.8318, 1.4356)$

Since the fixed point $(0, 1)$ is always a saddle point, we are just interested in the local stability of the other two fixed points. For these we evaluate the Jacobian matrix for each one of these points, and we obtain that the

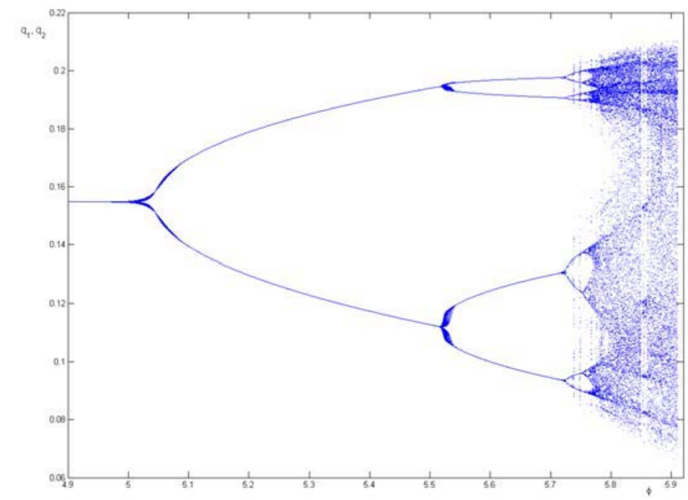

Figure 4: Bifurcation diagram for the player q_1 when $\alpha = 1.8$, $a = 0.9$, $b = 1.5$, $\beta = 1.2$, $v = 0.1$ and ϕ vary between 4.8 and 5.92

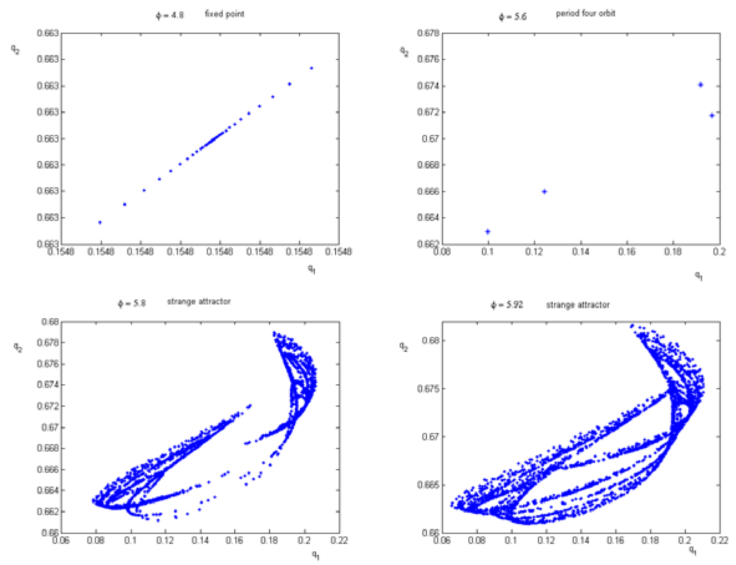

Figure 5: Periodic orbits and strange attractors for $\alpha = 1.8$, $a = 0.9$, $b = 1.5$, $\beta = 1.2$, $v = 0.1$ and when ϕ vary between 4.8 and 5.92

fixed point $(0.10967, 0.75411)$ is stable for $v = 4.8$ and loose stability at the first period-doubling bifurcation as can be seen in Figure 4. The other fixed point is always unstable. So, we can conclude that the only interesting equilibrium point is $(0.10967, 0.75411)$.

4 Conclusions

In this paper we investigate the dynamics of two nonlinear discrete-time maps, generated by duopoly games where the players can adopt different types of expectations in order to improve their gain. The study of the dynamics of these maps permit us to have information on the long-run behavior of the players.

When bounded rational and adaptive expectations are chosen, the nonlinear models becomes complicated and no analytical tool are available. For these reasons we proceeded to a detailed numerical analysis of the equilibria, and we found very complex dynamical behavior, from stable fixed points to chaotic attractors, through Neimark-Sacker and period-doubling bifurcations.

Although there exists multiple equilibria, boundary and Nash equililibra, few of the fixed points and periodic cycles present some interesting feature, the majority being unstable for any choice of the system parameters.

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