

Controlling Endogenous Cycles in an OLG Economy by the OGY Method

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Abstract

We show that very complicated dynamics arising, e.g. from an overlapping generations model (OLG) with production and an endogenous intertemporal decision between labour and leisure, which produces *hyperchaos* (both eigenvalues with modulus higher than 1), can in fact be controlled or managed with relative simplicity. The aperiodic and very complicated motion that stems from this model can be subject to control by very small perturbations in its parameters and turned into a stable steady state or into a regular cycle. Therefore, the system can be controlled without a change of its original properties. To perform the control of chaos in this economic model we apply the pole-placement technique, developed by Romeiras, Grebogi, Ott and Dayawansa (1992).

The application of control methods to chaotic economic dynamics may raise serious reservations, at least on mathematical and logical grounds, to some recent views on economics which have argued that economic policy becomes useless in the presence of chaotic motion (and thus, that the performance of the economic system cannot be improved by public intervention, i.e., that the amplitude of cycles cannot be controlled or reduced). In fact, the fine tuning of the system (that is, the control) can be performed without having to rely *only* on infinitesimal accuracy in the perturbation to the system, because the control can be performed with larger or smaller perturbations, but neither too large (because these would lead to a different fixed point of the system, therefore modifying its original nature), nor too small because the control becomes too inefficient.

Keywords: Nonlinear Economic Dynamics, Chaos in OLG Models, Control of Chaos

”Once we admit that an economy exists *in time*, that history flows from an irreversible past to an unknown future, the concept of equilibrium based on the analogy of a mechanic pendulum oscillating forth and back becomes unsustainable. *All traditional economics has to be radically changed*” (J. Robinson, 1973, 5, emphasis added)

”To prove the existence of chaos in any specific model is not a very easy task, but its presence may have vast consequences both for economic theory and policy. Just to mention two of them: if irregular fluctuations depend on the structure of the system, rather than on external disturbances, *intervention to eliminate or reduce them will have to change the system* rather than shield it from the shocks. Also if the behaviour of the system is extremely sensitive to changes in initial conditions (and therefore to shocks), as it is known to be the case in most types of chaos, *effective ‘fine tuning’ becomes impossible, unless policy measures are infinitely accurate.*” (A. Medio, 1987, 336, emphasis added)

1 Introduction

In the field of economics, as the above words of Medio and Robinson illustrate quite well, there is a feeling among many economists that the advent of chaos may produce a revolution in economic thought. This revolution would be the ultimate stage to overthrow the largely dominant practice within contemporary economics in academia and in policy-making institutions, which is based on the application of sophisticated mathematical tools to the analyses of economic theory and policy. It would be based, apparently, on three major points. *Firstly*, if modern economies are properly described as moving according to chaotic motion, then they seem to be almost impossible to understand, to predict, and to control using conventional or mathematical analytical methods. This would render conventional economic theory and policy totally irrelevant in contemporary economies. *Secondly*, any improvement on the functioning of these economies requires a radical change to their basic structure, which should be replaced by some alternative form apparently free from this type of motion. *Thirdly*, modern economics based on its frequent use of mathematical tools is missing the point because chaotic motion makes any accurate description of the true model-economy through the language of mathematics impossible, at least in the way this language has been used in the last forty years or so in the field of economics.

In order to find appropriate solutions to the unsolved mysteries in economic activity and to overcome the limitations of dominant analysis, some have

strongly argued that economics needs a radical new approach. This new approach is based on a "certain" view of chaos that seems to be crystallized by the popular allegory of the butterfly effect, in which a system that follows a chaotic motion is a process doomed to be *totally out of order, totally out of control* and, therefore, *completely unpredictable*. In fact, this view is not in contradiction with definitions of chaos that we may find in most dictionaries and encyclopedias, even those above any suspicion of low intellectual standards.

Although there might be some positive points in this new view of economics, in this paper we will argue that this view calling for a radical new approach seems to be rather simplistic and subject to serious reservations because it is based on a misconception of the fundamental characteristics of chaotic systems. This misconception arises because this new view apparently remains highly dependent upon the popular and romantic perspective of chaos that we have briefly highlighted above. It seems relatively superficial with respect to the fundamental characteristics of chaotic systems and overlooks the fact that in the last ten years or so there has been a remarkable amount of new results in the analysis of chaotic systems.

In this paper we use a standard model in economics — an overlapping generations model with no bequests, and no taxation¹ — to show that chaotic economic dynamics can be easily controlled even if the model exhibits some of the intricate forms of chaotic motion such as the one that has been named as hyperchaos (both Lyapunov exponents higher than zero). The introduction into this model of a constant relative risk aversion utility function and a linear Leontief technology leads to chaotic motion, and to endogenous business cycles of large amplitude as we shall see. The application of a small external perturbation to one accessible parameter of the model leaves the fundamental characteristics of the system unchanged — the fixed point that forms the basis of attraction remains the same — changes the fixed point from an unstable into a stable one, and eliminates those large business cycles (in other cases their amplitude may be reduced).

The control method that we are going to apply is a well known feedback control technique initially developed by Romeiras, Grebogi, Ott and Dayawansa for chaos control (1992). They made the very important observation that a chaotic attractor has embedded within it a dense set of unstable periodic orbits. Since they wish to make only very small perturbations to the system they do not envision creating new orbits with very different properties from the existing ones. Thus, they seek to exploit the dynamics of the already existing unstable periodic orbits. This method uses a linear approximation to the dynamics in the neighborhood of the desired periodic orbit, and consists in producing small perturbations to a system-wide accessible parameter to stabilize all unstable directions.

We are not aware of many papers in economics dealing with the process of controlling chaotic models despite an already impressive amount of work

¹Apart from the fact that transfer or bequests across generations would improve social welfare (which would vindicate public intervention), there is no other theoretical reason to justify the intervention of public agencies in the economic process in this model.

on the modelling side of economic chaotic dynamics.² Those which we have come across include Holyst *et al.* (1996), who studied a chaotic process of a dynamic game of two oligopolistic firms, Kopel (1997) analyzed the control of chaos in a model of evolutionary market dynamics, Kaas (1998) who applied control to a non-optimal conventional macroeconomic model, and Bala *et al.* (1998) who control chaos arising in the context of a tatonnement process of exchange economies. All these papers perform the control of chaotic motion using a different technique — the OGY method which is appropriate for the control of saddle point instability— and none of them controls the dynamics that has been named as hyperchaos.

This paper is organized as follows. In section 2 we present the basic characteristics of the overlapping generations model (OLG) with production and optimal leisure choice, a model that has been extensively studied as a source of generating chaotic motion. The model will have two basic characteristics: constant relative risk aversion in utility and a Leontief technology. In section 3, the dynamics of the OLG model is studied in great detail, including stable steady states, periodic motion, bifurcations and chaos. Section 4 deals with the control of chaotic motion, and the process of control is achieved using a relative risk aversion coefficient. Section 5 concludes.

2 An Overlapping Generations Model

In an interesting paper, Medio and Negroni (1996) used the basic OLG framework to study various combinations of utility and production functions that would lead to chaotic behavior. These combinations include CES *versus* Leontief (or fixed factor proportions) production functions and constant absolute *versus* constant relative risk aversion utility functions. In this paper we use a combination of a constant relative risk aversion (CRRA) utility function with a Leontief technology (L) — from which the model's shortname CRRAL stems — to show that chaotic economic dynamics can be easily subject to control. All other possible combinations of utility and production functions in the paper by Medio and Negroni also lead to chaos. However, as in the control of chaotic dynamics each specific dynamics may require a particular technique, in this paper we can only discuss one of those combinations due to shortage of space. For example, in Mendes and Mendes (2001) we apply a different technique to control the chaotic dynamics that arises from a constant absolute risk aversion with a Leontief technology (CARAL economy), which is based on the OGY method.³

We consider an overlapping generations model with production, where economic agents live for two periods (young at t , and old age at $t+1$), and in which there is an optimal intertemporal choice between labour and leisure: they work only in the first period and they consume in both periods. We also consider that there is a unique commodity in this economy which can be either consumed or

²For excellent surveys see, e.g., Benhabib (1992), Day (1999), Medio and Gallo (1995), Brock and Hommes (1997), Barnett *et al.* (1999), Lorenz (1993), and Boldrin *et al.* (2000).

³See Ott, Grebogi, and York (1990).

used in the production process as investment. Therefore, this economy has two major agents: firms that produce goods and services (by hiring labor and capital services) and maximize profits, and families which maximize utility and rent labour services in exchange for a wage rate.

Utility side. We will use the following designations: w_t for the real wage rate, R_{t+1} as the gross real interest rate, $u_1(c_t)$ is the utility of consumption in the first period, $u_2(c_{t+1})$ the utility of consumption in the second period, $v(l_t)$ the disutility of labour in the first period, s_t as the level of savings per person in the first period. We assume that the functions u_1, u_2, v are continuous and monotonously increasing on \mathbb{R}_+ , with u_1, u_2 as concave and v as convex on \mathbb{R}_+ .

The dynamic optimization problem can be written as

$$\max. \quad u_1(c_t) + u_2(c_{t+1}) - v(l_t) \quad (1)$$

s.t.

$$s_t \leq w_t l_t - c_t$$

$$c_{t+1} \leq R_{t+1} s_t$$

$$c_t, c_{t+1}, s_t, l_t > 0$$

Setting the Lagrangean for this optimal problem⁴

$$\mathcal{L} = u_1(c_t) + u_2(c_{t+1}) - v(l_t) + \lambda[(w_t l_t - c_t) R_{t+1} - c_{t+1}]$$

The optimal problem for each family can be determined by the first order conditions with respect to the three decision variables (c_t, c_{t+1}, l_t) and the multiplier

$$u_1'(c_t) - \lambda R_{t+1} = 0 \quad (2)$$

$$u_2'(c_{t+1}) - \lambda = 0 \quad (3)$$

$$v'(l_t) + \lambda w R_{t+1} = 0 \quad (4)$$

$$(w_t l_t - c_t) R_{t+1} - c_{t+1} = 0 \quad (5)$$

We shall proceed as follows to simplify this problem. Firstly, use equations (4) and (5) to eliminate w_t , and then obtain from the two first FOCs the result $R_{t+1} = u_1'(c_t) / u_2'(c_{t+1})$. Finally, substitute this result for R_{t+1} into the first step, and the optimal rule for intertemporal consumption and leisure over time will appear as

$$u_1'(c_t) c_t + u_2'(c_{t+1}) c_{t+1} - v'(l_t) l_t = 0 \quad (6)$$

Assuming constant relative risk aversion in all utility functions

$$u_1(c_t) = \frac{1}{\theta} c_t^\theta \quad 0 < \theta < 1$$

$$u_2(c_{t+1}) = \frac{1}{\alpha} c_{t+1}^\alpha, \quad 0 < \alpha < 1 \quad (7)$$

$$v(l_t) = \frac{1}{\gamma} l_t^\gamma, \quad \gamma > 1$$

⁴Note that the two constraints can in fact be reduced to only one by cancelling s_t .

the maximization of intertemporal utility in this CRRA framework leads to

$$c_{t+1} = (l_t^\gamma - c_t^\theta)^{1/\alpha}. \quad (8)$$

This is the first fundamental equation that characterizes the dynamics of this model, and represents the optimal evolution of consumption, derived from the consumer's intertemporal choice of consumption and leisure.

Technological side. To obtain the second equation of our dynamic system, we have to look at the technological side of the economy. We consider a linear Leontief production technology

$$y_t = \min[al_t, bk_{t-1}] \quad (9)$$

This equation assumes that output per person in period t , (y_t), is obtained by a linear combination of the amount of labour allocated to production in period t , (l_t), and the volume of capital accumulated in $t-1$, (k_{t-1}). The two parameters satisfy the following constraints: $a = 1$ for simplicity, and $b > 1$ for viability of capital accumulation.

The assumption of full employment and the restriction $a = 1$ lead to the result $y_t = l_t$. Moreover, the assumption of a constant capital/output ratio leads to $y_t = bk_{t-1}$. Taking into account the equilibrium condition in the product market, $y_t = k_t + c_t$, we obtain

$$y_t = b(y_{t-1} - c_{t-1}). \quad (10)$$

Moving forward one period, and using the result $y_t = l_t$, we can obtain the second fundamental equation that characterizes the dynamics of this OLG model

$$l_{t+1} = b(l_t - c_t), \quad b > 1. \quad (11)$$

Equations (8) and (11) represent the evolution of the system that is compatible with intertemporal optimization in constant relative risk aversion utility and equilibrium conditions in a Leontief economy.

3 The Dynamics of the CRRAL Economy

We have the following nonlinear 2-dimensional map which characterizes the overlapping generation model of a CRRAL economy

$$\begin{cases} c_{t+1} = (l_t^\gamma - c_t^\theta)^{1/\alpha} \\ l_{t+1} = b(l_t - c_t) \end{cases} \quad (12)$$

where $\gamma > 1$, $b > 1$, $0 < \alpha, \theta < 1$ are the parameters of the system. Despite its apparent simple form, the map presents an extremely complicated dynamic behavior. Different routes to chaos and lack of explicit analytical expressions for equilibria are noted for this map.

To compute the fixed points we have to solve the nonlinear system given by

$$\begin{cases} (l_t^\gamma - c_t^\theta)^{1/\alpha} = c_t \\ b(l_t - c_t) = l_t. \end{cases} \quad (13)$$

There are two fixed points: the first one is the trivial $E_1 = (0, 0)$, which is always locally unstable, and the second one is $E_2 = (c_* > 0, l_* > 1)$ which we cannot compute explicitly but it was shown by Medio and Negroni (1996) that is strictly positive.⁵ The equilibrium E_2 is stable if satisfies the following conditions

$$\begin{cases} 1 + Tr(J(E_2)) + Det(J(E_2)) > 0 \\ 1 - Tr(J(E_2)) + Det(J(E_2)) > 0 \\ 1 - Det(J(E_2)) > 0 \end{cases} \quad (14)$$

where $J(E_2)$ is the Jacobian matrix computed at the fixed point E_2

$$J(E_2) = \begin{bmatrix} -\frac{\theta}{\alpha}c^{\theta-\alpha} & \frac{\gamma}{\alpha}\left(\frac{b}{b-1}\right)^{\gamma-1}c^{\gamma-\alpha} \\ -b & b \end{bmatrix} \quad (15)$$

and $Tr(J(E_2))$ is the trace of the Jacobian matrix. This is a well known sufficient condition for the local stability giving the necessary and sufficient conditions for the two roots $\lambda_{1,2}$ of the characteristic equation to be inside the unit circle of the complex plane. Since

$$\begin{aligned} Tr(J(E_2)) &= -\frac{\theta}{\alpha}c^{\theta-\alpha} + b \\ Det(J(E_2)) &= (b-1)\frac{\gamma}{\alpha} + [\gamma(b-1) - b\theta]\frac{c^{\theta-\alpha}}{\alpha} \end{aligned} \quad (16)$$

the stability conditions (14) imply that the fixed point E_2 is stable if γ is sufficiently small, α is sufficiently large, or b is sufficiently small. In *Figure 1* we show the stability areas associated with the equilibrium E_2 where the curve F_1 denotes the condition $1 - Det(J(E_2)) = 0$ from relation (14), that is $\frac{\gamma}{\alpha} = \frac{1+b}{2(b-1)}$, and F_2 denotes the discriminant expression $(Tr(J(E_2)))^2 - 4Det(J(E_2)) = 0$, that is $\frac{\gamma}{\alpha} = \frac{(1+b)^2}{8(b-1)}$.

The purpose of this paper is to control chaotic orbits and, therefore, we should only be interested in the values of the parameters for which the map shows chaotic behavior. We will choose the parameter's values such that they are located somewhere in the black bullets region in *Figure 1* (complex conjugate eigenvalues, unstable equilibrium), that is for $\frac{\gamma}{\alpha} > \frac{1+b}{2(b-1)}$.

⁵In the paper of Medio and Negroni, the fixed point E_2 is defined for $(c_* > 1, l_* > 1)$, when actually almost all of the values of the c coordinate computed here lie in the interval $]0, 1[$.

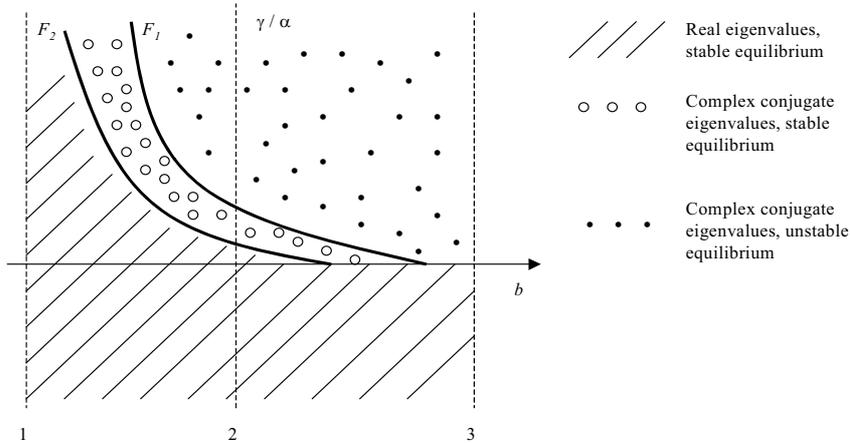


Figure 1: *Stability of the equilibrium E_2*

As we are dealing with a nonlinear 2-dimensional map, and as the theoretical tools to prove the existence of chaotic motion in 2-D are still very poor, we do this resorting to computer numerical approximations. Without loss of generality we fix $\alpha = \theta = 0.2, b = 1.2$ and assume that γ can vary. We consider the following initial conditions $(c_0, l_0) = (0.1, 1.16)$ situated in the basin of attraction and we start to study the behavior of the system when the parameter γ is varied in the interval $]1, 1.73[$.

Generically, when the control parameter varies, a periodic solution can lose stability through various types of bifurcations and the resulting solution depends on how the multipliers leave the unit circle. Recalling *Figure 1*, we realize that starting from a stable configuration and increasing γ we have to pass through the boundaries delimited by F_1 and F_2 , and so, loss of stability (boundary F_1) takes place through a Neimark-Sacker bifurcation. The Neimark-Sacker bifurcation (or secondary Hopf bifurcation) is a local bifurcation which produces a qualitative change in some neighborhood of the fixed point when a pair of complex conjugate eigenvalues leaves the unit circle away from the real axis and, as a consequence, an invariant closed curve (circle) is bifurcating for some value of γ , say γ_* , around the fixed point. This (unique) invariant circle occurs if certain, rather general, nonresonance conditions hold for the normal form of the system. We assume that these conditions ⁶ are satisfied here and for a rigorous proof

⁶A Neimark-Sacker bifurcation or Hopf bifurcation for maps is characterized by the following: given a parameter dependent map on $\mathbb{R}^n, n \geq 2, x \mapsto A(a)x + G(x, a)$ with

(N1) G is a C^k -smooth mapping, $k \geq 2$, from $\mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}^n, G(0, a) = 0, G_x(0, a) = 0, a \in \mathbb{R}$

(N2) $A(a^*)$ has a complex conjugate pair of eigenvalues with modulus 1, i.e., $|\lambda_{1,2}| = 1$, while all other eigenvalues have modulus strictly less than one

(N3) $r'(a^*) \neq 0$, where $r(a)$ is the modulus of the branch of eigenvalues with $r(a^*) = 1$.

of existence of Neimark-Sacker bifurcations for generic 2-dimensional maps, we refer to Kuznetsov (1998).

Along this process we numerically encounter a multitorus (or torus break-down) route to chaos. In this route to chaos, a torus attractor bifurcates into periodic orbits of consecutively increasing (decreasing) periods, i.e., windows of quasi periodic and periodic behavior appear alternately as the parameter γ is changed. After several such bifurcations, a periodic orbit finally bifurcates into a chaotic attractor. These various results will be discussed and illustrated by a large set of figures in the remaining part of this section.

The Neimark-Sacker bifurcation takes place for $\gamma = 1.1$ and for values lower than $\gamma = 1.1$ a stable equilibrium exists. We start with an initial value of the control parameter, let us say $\gamma = 1.098$. As the analysis above indicates this value leads to a stable equilibrium point, and the attractor of this type of equilibrium is represented graphically in *Figure 2*. Both variables of the dynamic system (c_t and l_t , respectively, consumption and labour services per person) converge towards a unique and stable point independently from the initial state of the economy. The equilibrium point is characterized by the optimal intertemporal values $c_* = 0.2419$, $l_* = 1.4518$. The eigenvalues of the Jacobian matrix computed at the equilibrium point are $\lambda_{1,2} = 0.0999 \pm 0.9929i$ with $|\lambda_{1,2}| \cong 0.9960$. This dynamic process can also be represented by the time series of both variables, which are also shown in *Figure 2*. As we can easily see both converge steadily and cyclically towards their steady state values.

As we have already mentioned the Neimark-Sacker bifurcation takes place for $\gamma = 1.10$, bifurcation value obtained from the stability condition $\frac{\gamma}{\alpha} = \frac{1+b}{2(b-1)}$. For this parameter value, the equilibrium point occurs at $c_* = 0.2417$, $l_* = 1.4506$ and the associated pair of complex conjugate eigenvalues are $\lambda = 0.1000 \pm 0.9949i$ with $|\lambda| \cong 1.0000$, which shows that varying the parameter γ from 1.098 to 1.1, the eigenvalues are approximating the unit circle and the equilibrium is changing its stability properties through a Neimark-Sacker bifurcation. *Figure 3* illustrates the phase plot and the coordinates time series for the bifurcation value of γ .

Continuing to increase the value of γ , we see what happens for $\gamma = 1.11$. The coordinates of the equilibrium are $c_* = 0.2407$, $l_* = 1.4447$ and the associated eigenvalues are $\lambda = 0.0999 \pm 1.0049i$. The modulus of the complex conjugate eigenvalues is $|\lambda| \cong 1.020$, and so we can conclude that the equilibrium became unstable and an invariant closed curve was created around the fixed point, which is shown in the *Figure 4* together with the time series of the two state variables in this control problem. As one would expect from a quasi periodic motion, the phase plot is in the form of a smooth closed curve (the cross section of a two-torus).

As γ is further increased, however, the phase plot starts to fold and — interrupted by periodic windows — a quasi periodic transition to chaos takes

Under the above hypotheses the map has an invariant closed curve of radius $O(\sqrt{|a^* - a|})$ surrounding the origin for all a in a one-side neighborhood of a^* . The closed curve is attracting (repelling) if zero is an asymptotically stable (unstable) fixed point of the map at $a = a^*$.

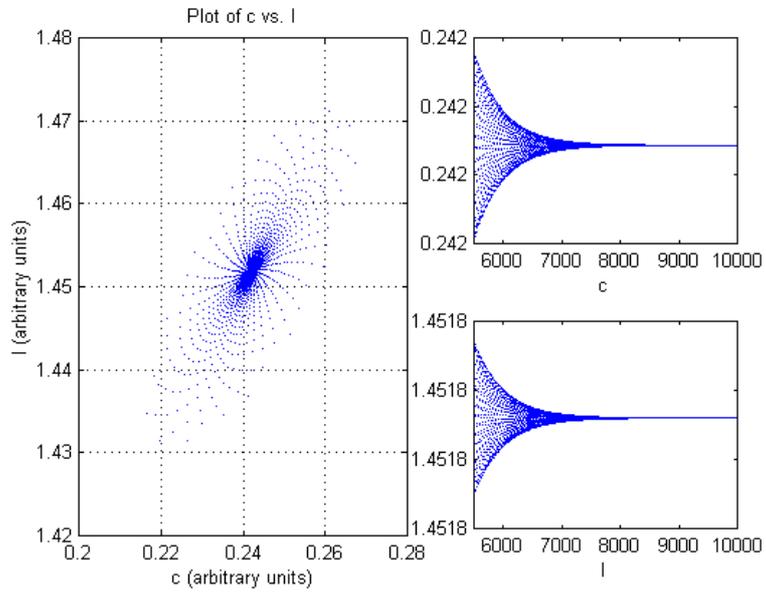


Figure 2: $\gamma = 1.098$, the stable fixed point before the Neimark-Sacker bifurcation occurs.

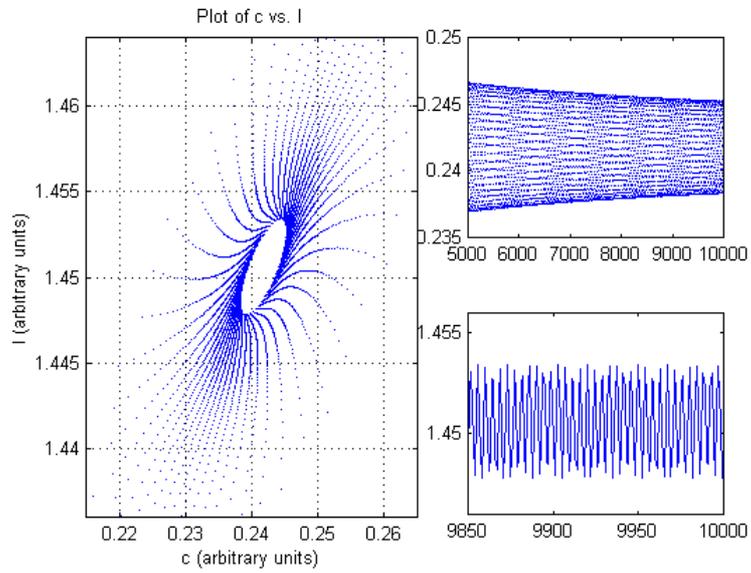


Figure 3: $\gamma = 1.10$, the Neimark-Sacker bifurcation

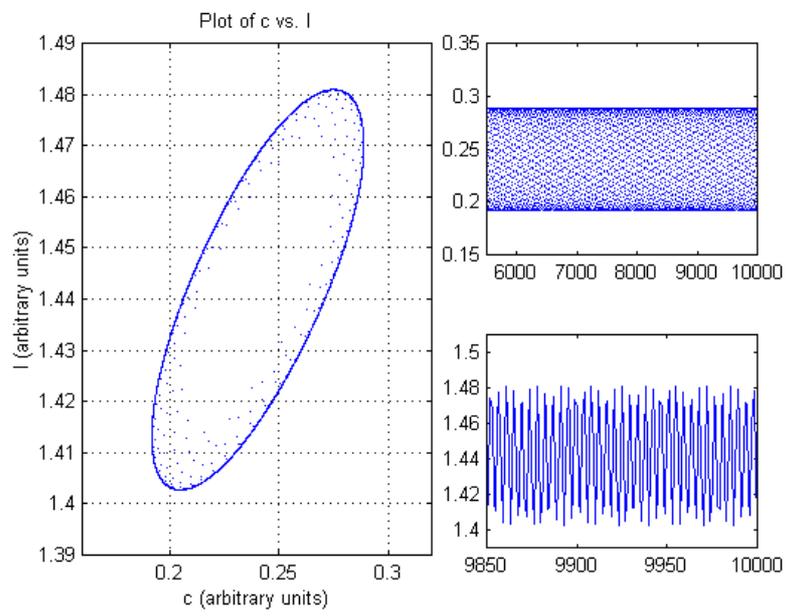


Figure 4: $\gamma = 1.11$, the stable invariant closed curve around the fixed point created after the bifurcation

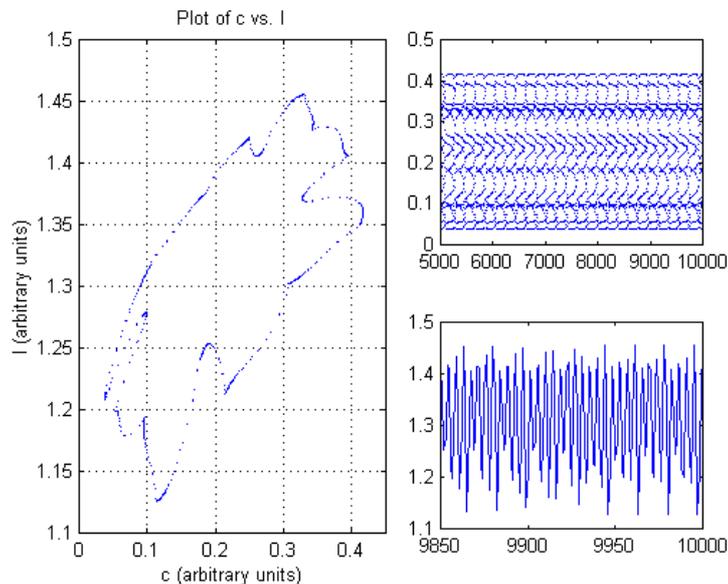


Figure 5: $\gamma = 1.316$, the breakdown of the invariant closed curve

place. We can see that the "circle" after being stretched, shrunk and folded creates a new phenomena: the breakdown of the invariant closed curve (see for instance *Figure 5*), which leads to the appearance of various invariant closed curves. Looking at the time series of the state coordinates, we can observe an expansion in the windows associated with each of the two sequences.

For $\gamma > 1.32$ we obtain multiple invariant closed curves brought by Neimark-Sacker bifurcations of iterates of the original map. In these cases, the dynamics from one circle to another are periodic (and thus easily predictable), but the dynamics on each closed curve, may be periodic or quasi periodic. Moreover, these closed curves may break, leading to multiple fractal tori on which the dynamics are chaotic. Following the Neimark-Sacker bifurcations, quasiperiodic solutions with windows of frequency locking appear. The radius of the quasiperiodic solution grows as γ is further increased. *Figure 6* represents seventeen invariant closed curves brought about by a Neimark-Sacker bifurcation of the 17th iterate of the map of the system, obtained for $\gamma = 1.362$. Following our procedure, we also present the time series associated with each of the coordinates which shows very clearly the quasi periodicity of the orbits.

The bifurcation diagram of the l coordinate, also shows all the remarkable phenomena that we have been describing (see for instance *Figure 7*). It is easy to realize that the stable fixed point bifurcates into an invariant closed curve (dense region in the bifurcation diagram), and after two windows (the first between 1.32 and 1.37, and the second between 1.46 and 1.5) where periodic

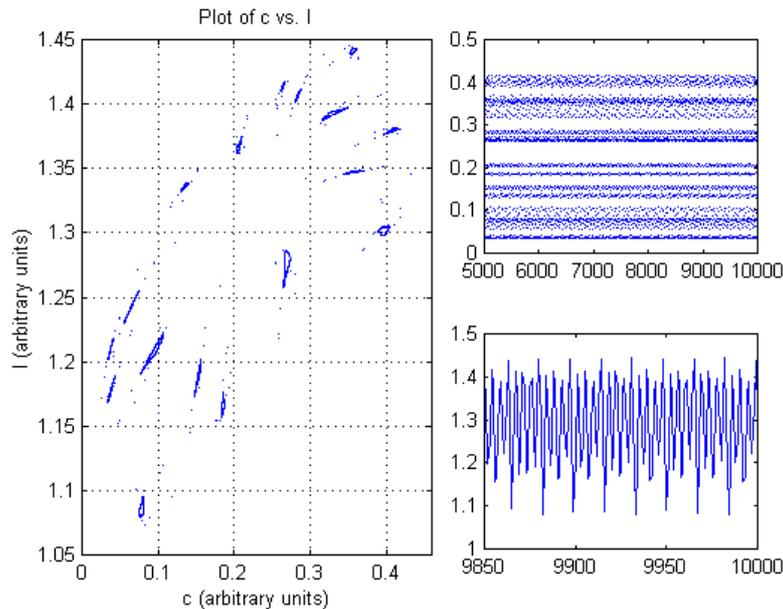


Figure 6: $\gamma = 1.362$, the existence of multiple invariant closed curves

orbits bifurcate once again through Neimark-Sacker bifurcations into multiple invariant closed curves, chaotic motion is finally obtained.

A strange attractor is produced by successive stretching and folding. The attractor in *Figure 8* is a bounded region in the phase space to which all sufficiently close trajectories are asymptotically attracted for a long enough period of time. While individual trajectories are chaotic, the chaotic attractor reveals information about the long-term trends of the system. The stretching causes orbits on the attractor to exhibit sensitive dependence on initial conditions (chaos) and the folding causes the fractal (strange) structure. The impressive structure appearing for $\gamma = 1.38$ is chaotic and is represented with the associated time series of the coordinates in *Figure 8*. The time series change from the quasi-periodic shape that we have already encountered in previous simulations to a totally "random" shape. The equilibrium is $c_* = 0.22134, l_* = 1.32806$ and the eigenvalues are $\lambda_{1,2} = 0.1000 \pm 1.2449i$ with $|\lambda_{1,2}| \cong 1.559$. The strange attractor is produced by the breaking of the invariant circles and the appearance of the seventeen chaotic regions changes as they are linked into a single chaotic attractor. Full developed chaos occurs also for $\gamma = 1.68$, after passing the second window of multiple Neimark-Sacker bifurcations for the iterates of the original map (see *Figure 9*). The fractal structure of the attractors is evident in both cases.

Lyapunov exponents are a dynamic measure of chaos that average the sepa-

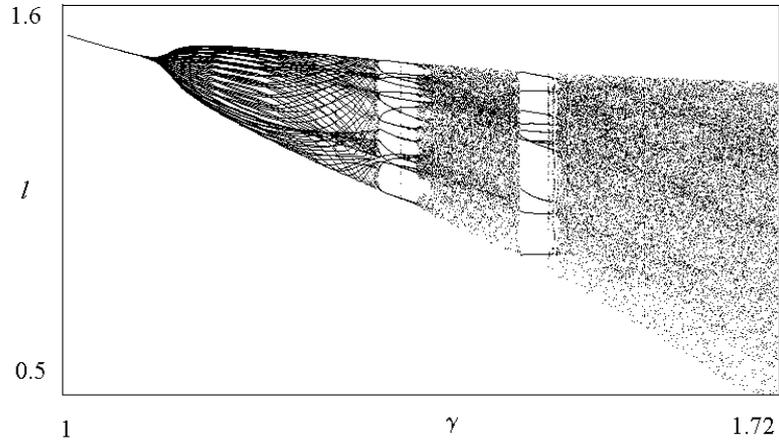


Figure 7: *Bifurcation diagram for the l state coordinate*

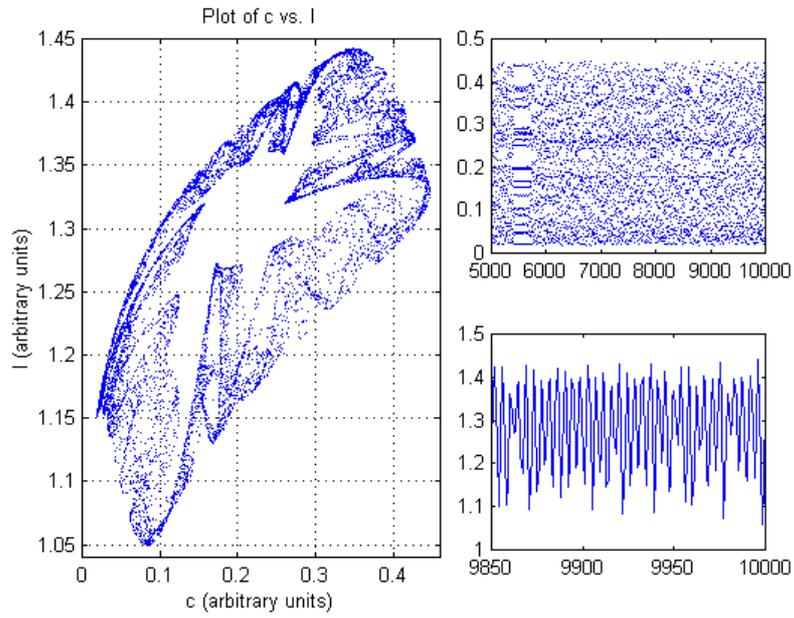


Figure 8: $\gamma = 1.38$, *chaotic trajectory*

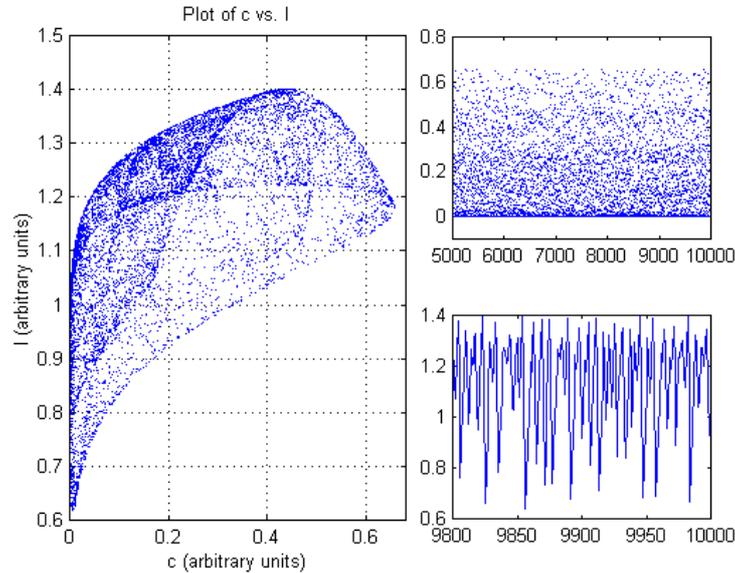


Figure 9: $\gamma = 1.68$ full developed chaos

ration of the orbits of nearby initial conditions as the system moves forward in time. The chaotic attractor displayed in *Figure 9* has both Lyapunov exponents positive, computed to be approximately equal to 0.19 and 0.13 respectively. These Lyapunov exponents are represented in *Figure 10*. They are both larger than zero and the complex conjugate eigenvalues are situated outside the unit circle which indicates that we are actually dealing with a very complex phenomenon known as *hyperchaos*.

4 Controlling Chaotic Economic Motion

In principle the control of chaotic systems does not differ from the control of general nonlinear systems. However, there is a substantial difference which was elegantly summarized by Mees (1998): "if we ask the right sort of questions, questions that may differ from those normally asked by a control theorist, we may be able to get a chaotic system to do something desirable *with rather little control effort on our part*. I call this *control by smart butterflies*, because of the infamous butterfly effect, which says that chaotic systems are sensitive to small changes" (1998, vii). That is, conventional classical control techniques control the dynamics of nonlinear processes through the use of brute force, having in fact frequently to change the nature of the very system that is subject to control because these systems are not sensitive to small changes in their parameters. However, in the case of chaotic systems, as these are sensitive to very small

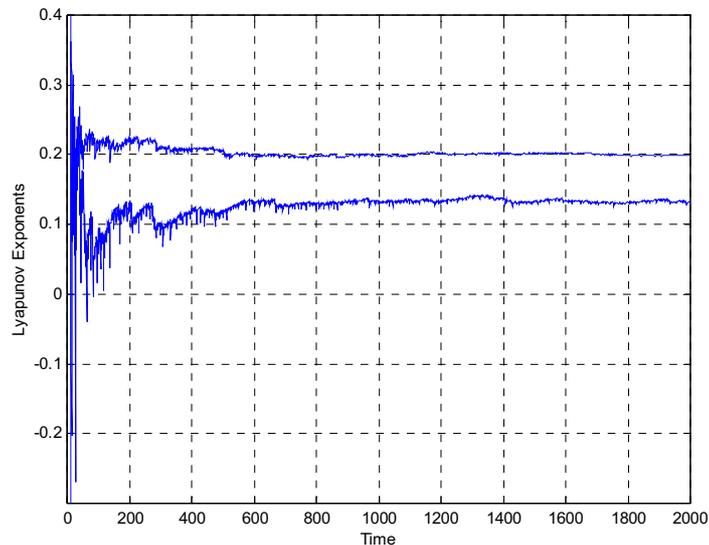


Figure 10: *Convergence of the two positive Lyapunov exponents.*

changes in the parameters, a small butterfly effect in one of them is (in most cases) all that is required to control their outcome, without changing the very nature of the controlled system in any relevant way. In short, if a chaotic system has an unstable fixed point, the control procedure turns this unstable point into a stable one, by a very small perturbation, leaving the system's initial fundamental characteristics untouched.

In general, the techniques for feedback control of chaos presented thus far in the literature have some common features which we will briefly summarize. The control is usually designed for parameter values where the system is known to exhibit chaotic motion, and is typically of the form $u = u(x - x_*)$ where x is the system state vector, and x_* is an unstable equilibrium of interest, which lies on a chaotic attractor. The control function u is not necessarily smooth. Thus, when an input is altered on the basis of the difference between the actual output of the system and the desired output, the system is said to involve feedback. Note that x_* can also be a periodic orbit. The Ott-Yorke-Grebogi method (OGY method) (1990) and the pole-placement technique (see Ogata (1997) and Romeiras *et al.* (1992)) belong to feedback control. The pole-placement method extends that of OGY, allowing for a more general choice of the feedback matrix and implementation to higher-dimensional systems.⁷

In what follows we will apply the pole-placement method to the CRRAL eco-

⁷See Mendes and Mendes (2001) for control of chaotic motion in a OLG economic model with the OGY method.

conomic model that we have been discussing along this paper and we stabilize an unstable period-one orbit embedded in the chaotic attractor. By applying small, adequate chosen temporal perturbations to an accessible control parameter of the dynamical system, the original chaotic trajectory can be converted into the desired fixed point. The control parameter that we will use is the disutility of labor relative risk aversion coefficient, γ .

4.1 Controlling through γ by pole-placement technique

It was shown numerically in the previous section that for $\gamma = 1.38$, $b = 1.2$, $\alpha = \theta = 0.2$ the system exhibits a chaotic attractor (*Figure 8*). We fix these parameter values and consider that γ is the control parameter which is available for external adjustment but is restricted to lying in some small interval $|\gamma - \gamma_0| < \delta, \delta > 0$ around the nominal value $\gamma_0 = 1.38$. The system becomes:

$$\begin{cases} f : c_{t+1} = (l_t^\gamma - c_t^{0.2})^{1/0.2} \\ g : l_{t+1} = 1.2(l_t - c_t) \end{cases} \quad (17)$$

We vary the control parameter γ with time t in such a way that for almost all initial conditions in the basin of the chaotic attractor, the dynamics of the system converge onto a desired time periodic orbit contained in the attractor. The control strategy is the following: we find a stabilizing local feedback control law which is defined in a neighborhood of the desired periodic orbit. This is done by considering the first order approximation of the system at the chosen unstable periodic orbit. The ergodic nature of the chaotic dynamics ensures that the state trajectory eventually enters into the neighborhood. Once inside the neighborhood, we apply the stabilizing feedback control law in order to steer the trajectory towards the desired orbit.

In this case we consider the stabilization of the unstable period-one orbit $E_2 : (c_*, l_*) = (0.2213, 1.3280)$. The map can be approximated in the neighborhood of the fixed point by the following linear map,

$$\begin{bmatrix} c_{t+1} - c_* \\ l_{t+1} - l_* \end{bmatrix} \cong A \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} + B[\gamma - \gamma_0] \quad (18)$$

where

$$A_{(2 \times 2)} = \begin{bmatrix} \frac{\partial f(c_*, l_*)}{\partial c_t} & \frac{\partial f(c_*, l_*)}{\partial l_t} \\ \frac{\partial g(c_*, l_*)}{\partial c_t} & \frac{\partial g(c_*, l_*)}{\partial l_t} \end{bmatrix} \quad \text{and} \quad (19)$$

$$B_{(2 \times 1)} = \begin{bmatrix} \frac{\partial f(c_*, l_*)}{\partial \gamma} \\ \frac{\partial g(c_*, l_*)}{\partial \gamma} \end{bmatrix} \quad (20)$$

are the Jacobian matrixes with respect to the control state coordinates (c_t, l_t) and to the control parameter γ . The partial derivatives are evaluated at the nominal value γ_0 and at (c_*, l_*) . In our case we get

$$\begin{bmatrix} c_{t+1} - 0.22 \\ l_{t+1} - 1.32 \end{bmatrix} \cong \begin{bmatrix} -1.0 & 2.29 \\ -1.2 & 1.2 \end{bmatrix} \begin{bmatrix} c_t - 0.22 \\ l_t - 1.32 \end{bmatrix} + \begin{bmatrix} 0.62 \\ 0 \end{bmatrix} [\gamma - 1.38] \quad (21)$$

Next, we check whether the system is controllable. A controllable system is one for which a matrix $H_{(1 \times n)}$ can be found such that $A - BH$ has any desired eigenvalues. This is possible if $\text{rank}(C) = n$, where n is the dimension of the state space, and

$$C = [B : AB : A^2B : \dots : A^{n-1}B]. \quad (22)$$

In our case it follows that

$$C = [B : AB] = \begin{bmatrix} 0.62 & -0.62 \\ 0 & -0.75 \end{bmatrix} \quad (23)$$

which obviously has rank 2, and so we are dealing with a controllable system.

If we assume a linear feedback rule (control) for the parameter γ of the form

$$[\gamma - \gamma_0] = -H \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \quad (24)$$

where $H_{(1 \times 2)} := [h_1 \quad h_2]$, then the linearized map becomes

$$\begin{bmatrix} c_{t+1} - c_* \\ l_{t+1} - l_* \end{bmatrix} \cong [A - BH] \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \quad (25)$$

that is

$$\begin{bmatrix} c_{t+1} - 0.22 \\ l_{t+1} - 1.32 \end{bmatrix} \cong \begin{bmatrix} -1.0 - 0.62h_1 & 2.29 - 0.62h_2 \\ -1.2 & 1.2 \end{bmatrix} \begin{bmatrix} c_t - 0.22 \\ l_t - 1.32 \end{bmatrix} \quad (26)$$

which shows that the fixed point will be stable provided that the (2×2) -matrix $A - BH$ is asymptotically stable, that is, all its eigenvalues have modulus smaller than one. The eigenvalues μ_1, μ_2 of the matrix $A - BH$ are called the "regulator poles" and the problem of placing these poles at the desired location by choosing H with A, B given is the "pole-placement problem". If the controllability matrix C from equation (22) is of rank n , $n = 2$ in our case, then the pole-placement problem has a unique solution. This solution is given by

$$H = [\alpha_2 - a_2 \quad \alpha_1 - a_1]T^{-1} \quad (27)$$

where $T = CW$ and

$$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -0.20 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here $\{a_1, a_2\}$ are the coefficients of the characteristic polynomial of A , i.e.,

$$\begin{aligned} |A - \lambda I| &= \lambda^2 + a_1\lambda + a_2 = \lambda^2 - 0.20\lambda + 1.559 \\ \Rightarrow a_1 &= -(\lambda_1 + \lambda_2) = -0.20, \quad a_2 = \lambda_1\lambda_2 = 1.559 \end{aligned}$$

and $\{\alpha_1, \alpha_2\}$ are the coefficients of the desired characteristic polynomial of $A - BH$, i.e.,

$$\begin{aligned} |(A - BH) - \mu I| &= \mu^2 + \alpha_1\mu + \alpha_2 \\ \Rightarrow \alpha_1 &= -(\mu_1 + \mu_2) \\ \Rightarrow \alpha_2 &= \mu_1\mu_2 \end{aligned}$$

From equation (27) we get that

$$\begin{aligned} H &= [\mu_1\mu_2 - 1.559 \quad -(\mu_1 + \mu_2) + 0.20] \begin{bmatrix} 0 & -1.32 \\ 1.59 & -1.59 \end{bmatrix} \\ &= [-1.59(\mu_1 + \mu_2) + 0.318 \quad -1.32\mu_1\mu_2 + 1.59(\mu_1 + \mu_2) + 1.752]. \end{aligned}$$

Since the 2-D map is nonlinear, the application of linear control theory will succeed only in a sufficiently small neighborhood U around (c_*, l_*) . Taking into account the maximum allowed deviation from the nominal control parameter γ_0 and equation (24), we obtain that we are restricted to the following domain

$$S_H = \left\{ (c_t, l_t) \in \mathbb{R}^2 : \left| H \begin{bmatrix} c_t - c_* \\ l_t - l_* \end{bmatrix} \right| \leq \delta \right\}. \quad (28)$$

This defines a slab of width $2\delta/|H|$ and thus we activate the control (24) only for values of (c_t, l_t) inside this slab, and choose to leave the control parameter at its nominal value when (c_t, l_t) is outside the slab.

Any choice of regulator poles inside the unit circle serves our purpose. There are many possible choices of the matrix H . In particular, it is very reasonable to choose all the desired eigenvalues to be equal to zero and in this way the target would be reached at least after n periods, and, therefore, a stable periodic orbit is obtained out of the chaotic evolution of the dynamics.

The time efficiency in the control process is another issue that can also be considered. Romeiras *et al.* (1992) investigated numerically which choice of the feedback matrix H exhibits the shortest stabilization time. Because the linearized equation (25) does not take into account any nonlinearity that is part of the original chaotic system, the control may not be able to bring the orbit

to the fixed point, despite the fact that it is already in the slab. In this case, the orbit will leave the slab and continue to wander chaotically as if there were no control. Since the orbit on the uncontrolled chaotic attractor is ergodic, at some time it will eventually satisfy the condition (28) and also be sufficiently close to the desired fixed point so that control is achieved. Thus, a stable orbit is created, which for a typical random initial condition, is preceded by a chaotic transient in which the orbit is similar to other orbits on the uncontrolled chaotic attractor. The length τ of such chaotic transient depends sensitively on the initial conditions of the particular orbit. For initial conditions randomly chosen in the basin of attraction, the distribution of chaotic transient lengths is exponential

$$\phi(\tau) = \frac{1}{\langle \tau \rangle} \exp\left(-\frac{\tau}{\langle \tau \rangle}\right)$$

for large τ . The quantity $\langle \tau \rangle$ is the characteristic length of chaotic transient, called in the present case the average time to achieve control.

The transient phase where the orbit wanders chaotically before control being applied can be shortened by applying the targeting technique proposed by Shinbrot *et al.* (1990). It was pointed out that orbits can be rapidly brought to a target region on the attractor by using small control perturbations when the orbit is far from the neighborhood of the periodic orbit to be stabilized. The idea is that, since chaotic systems are exponentially sensitive to perturbations, after some time these perturbations produce a large effect on the orbit location and can be used to guide it.

4.2 Numerical examples

Two cases will be illustrated. Firstly, we consider different values of h_1 and h_2 and fix a certain orbit initiated at a particular point in the basin of attraction. As we will see, the controlled orbit will converge towards the fixed point but takes different periods of time in order to fully accomplish that convergence, depending on the values of h_1 and h_2 . Secondly, the chaotic trajectory will also converge to the fixed point if, in contrast, we consider fixed values of h_1 and h_2 and *randomly* choose some initial conditions. In all examples we iterate the system for 100 iterations until the chaotic behavior is perfectly evident and the iterates are distributed over the attractor and then apply the control strategy when the orbit is inside the slab. After this time point the system is forced to follow the desired orbit.

In *Figure 11* we show the time series of the chaotic trajectory initiated at the point $(c_0, l_0) = (0.320, 1.427)$ which we have chosen to control. In contrast, *Figure 12* presents the controlled orbit converging to the stabilized fixed point when the feedback matrix H is chosen such that the eigenvalues of $(A - BH)$ are $\mu_1 = \mu_2 = 0$. This implies that $\mu_1 + \mu_2 = 0$, $\mu_1\mu_2 = 0$ and so $H = [0.318 \ 1.752]$. For this control strategy we have also chosen $\delta = 0.1$.

For $H = [-1.28 \ 3.01]$ the motion will converge to the stable orbit of period one which is shown in *Figure 13*. The matrix $A - BH$ has a pair of real eigenval-

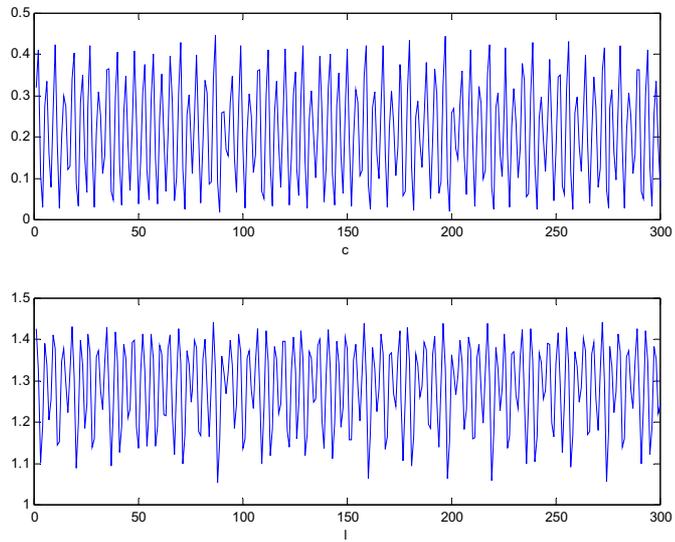


Figure 11: *Original chaotic orbit*

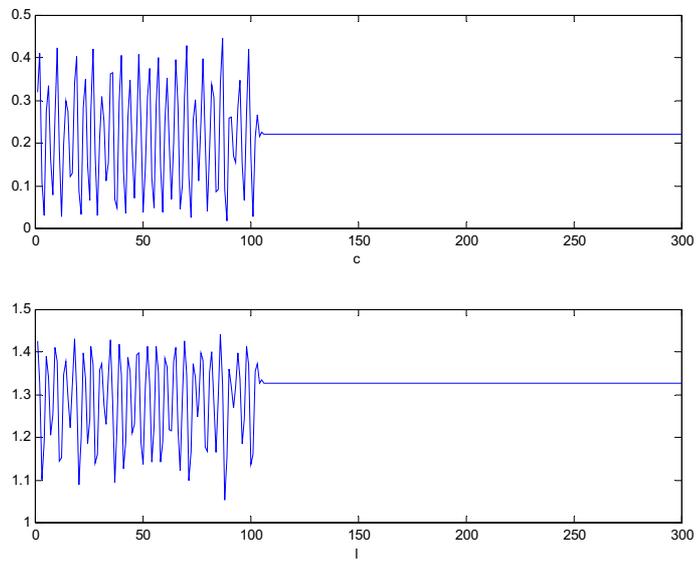


Figure 12: *Controlled chaotic orbit for $\delta = 0.01$ and $H = [0.3184 \ 1.7528]$*

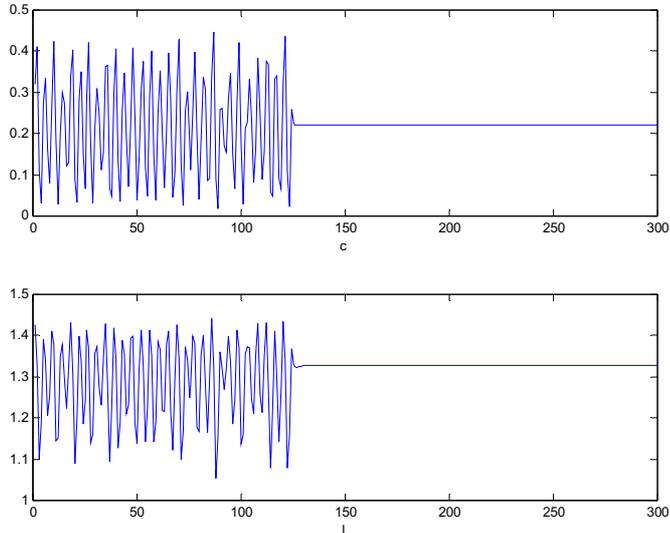


Figure 13: *Controlled orbit for $\delta = 0.1$ and $H = [-1.28 \ 3.01]$*

ues equal to $\mu_1 = \mu_2 = 1.2$. The linear control is activated for these values and for the time index 100. After switching on the control, the orbit continues to exhibit chaotic behavior for some time, unchanged from the uncontrolled case, because it is not close enough to the fixed point. After some steps, this is eliminated and the orbit is rapidly brought to the fixed point. We can observe that in this case, the orbit to enter the slab and the control to be achieved both will take a longer time span to be accomplished in comparison to the first example.

In what follows, we will place the poles such that $H = [0.318 \ 1.752]$ and will consider randomly chosen initial conditions. *Figures 14, 15* show clearly that the chaotic transient depends sensitively on the initial conditions of the particular orbits.

For $\delta = 0.1$ and randomly chosen initial conditions, the pole placement control strategy works very well for this system. Exhaustive numerical experimentations show that almost all initial conditions lead to controllable orbits and the time to achieve the control is no longer than 100 iterates. For $\gamma = 1.68$, $b = 1.2$, $\alpha = \theta = 0.2$ analogous control results were obtained.

Further numerical experimentations were done. For an external adjustment such that the interval in which the control parameter can vary is smaller than in the previous examples, e.g., for $\delta = 0.01$, and for randomly chosen initial conditions, the control strategy still works very well. However, as one would expect the time span needed to stabilize the randomly chosen trajectory is much larger because of the smaller value for δ . In *Figure 16* we show that it

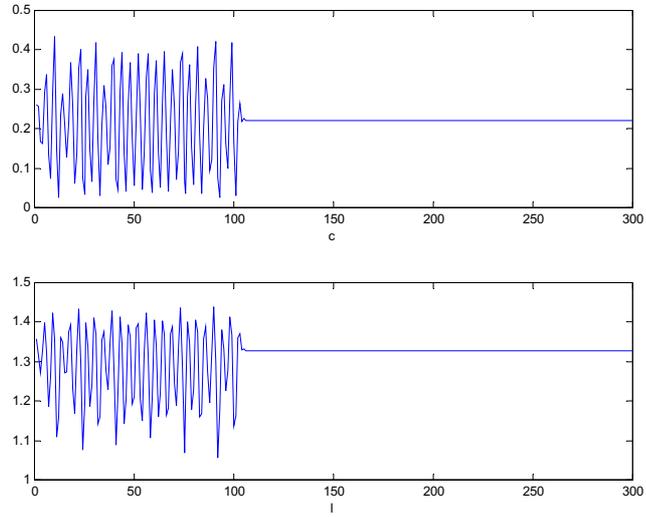


Figure 14: *Control of a randomly chosen trajectory for $\delta = 0.1$.*

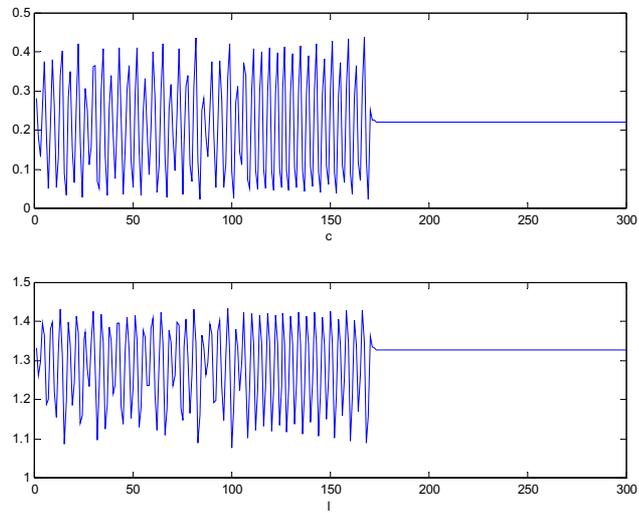


Figure 15: *Control of a randomly chosen trajectory for $\delta = 0.1$.*

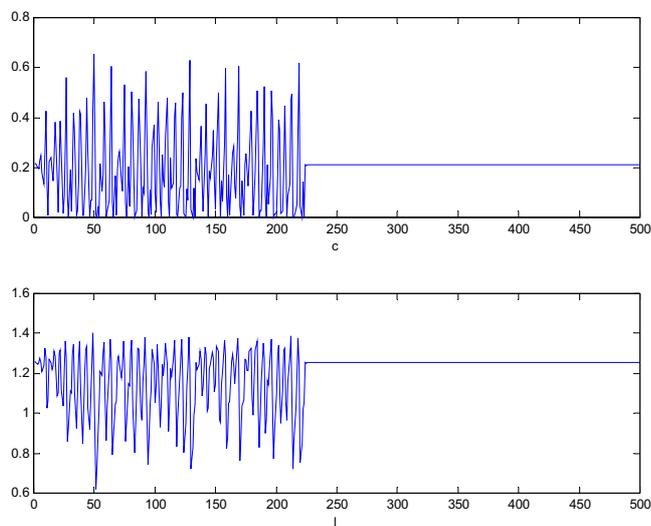


Figure 16: *Control of a randomly chosen trajectory for $\delta = 0.01$.*

takes around 125 periods to have a randomly chosen trajectory under control for this case. However, for other initial conditions, the time span required to achieve control becomes much larger.

5 Concluding Remarks

We have applied the pole-placement control technique to an overlapping generations model with production and an endogenous intertemporal decision between labour and leisure. It was shown that the aperiodic and complicated motion that arises from the dynamics of the model can be easily subject to control by small perturbations in its parameters and be turned into a stable steady state. This simple exercise may raise serious reservations to the recent views sustaining that economic policy becomes impossible or useless in the presence of chaotic motion, at least on purely logical and mathematical grounds.

Two major points should be stressed due to their potential relevance for economics. Firstly, and contrary to the view of Medio in the opening quotation, the fine tuning of the system (that is, the control) can in fact be performed without having to rely *only* on infinitesimal accuracy in the perturbation process, because the control can be performed with larger or smaller perturbations, but neither too large (because these would change the chaotic initial fixed point, modifying therefore the nature of the system), nor too small because the control becomes too inefficient. In this paper we assumed that γ was the control

parameter which was available for external adjustment and was restricted to lie in some small interval $|\gamma - \gamma_0| < \delta$, $\delta > 0$ around the nominal value $\gamma_0 = 1.38$. Two values for δ were considered: $\delta = 0.1$ and a much smaller interval $\delta = 0.01$. In both cases the control was easily achieved using either randomly or arbitrarily chosen initial conditions.

Secondly, the fundamental characteristics of the model are not changed by the control procedure — as the fixed point that forms the basis of attraction remains the same — all that is changed is its stability properties, from unstable to a stable one — and the large cycles are eliminated. Therefore, instead of rendering economic policy useless, chaotic motion may in fact even vindicate the intervention of public agencies in models where such intervention would not be largely justified in accordance with conventional theory.

In our opinion, these seem relevant arguments to question the view which sustains that if modern economies are properly described as moving according to chaotic motion, then they seem to be almost impossible to understand, to predict, and to control by using conventional analytical or mathematical methods. This view would render conventional economic theory and policy totally irrelevant in contemporary economies. Moreover, the possibility of obtaining relevant positive knowledge of chaotic economic systems (that is, creating abstract mathematical models and testing them on empirical and numerical grounds) is not strictly confined to the remarkable result that arises from controlling chaotic dynamics as we show in this paper. Research in the predictive power of non linear time series also confirm this possibility.

Therefore, contrary to the view of Medio and Robinson presented in the past, this means that if economic reality behaves in accordance with the laws of chaos, there still seems to exist plenty of room for improvement of the system's performance, and for relative stability through the implementation of economic policy, without having to throw away the very nature of the system. In fact, recent developments in the field of chaotic systems have made clear that many chaotic dynamics can be largely understood by rational and mathematical tools (in terms of the nature of the processes that are behind these dynamics), can be controlled and can be predicted as well.

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