

Active Interest Rate Rules and the Role of Stabilization Policy

Vivaldo M. Mendes
Diana A. Mendes

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Second draft, September – 2006

Abstract

In a series of papers, Benhabib, Schmitt-Grohé and Uribe (2001a, 2001b, 2001c, 2002 and 2004) have shown that active interest rules may lead to very unexpected consequences: indeterminacy, deflation traps, large cyclical instability, and can even lead to chaotic dynamics under standard sets of parameter values. This paper explores this particular model and puts forward four basic points: (i) the model developed by Benhabib and associates seems to suffer from serious drawbacks to be used as a theoretical benchmark to guide optimal monetary policy, as the more aggressive the central bank becomes, the more unstable the economy will be; (ii) the time span required to achieve successful control is generally small, by linear feedback techniques — the OGY method — without producing modifications to the original model, apart from locally changing its type of stability; (iii) ignorance about the true state of initial conditions are not a serious impediment to obtain control of the chaotic dynamics in the model; (iv) we argue that the conventional view of economic policy in nonlinear general equilibrium models — when endogenous fluctuations exist in optimizing models, the associated policy advice is *laissez-faire* — seems to be based on a misconception of chaos in general, and on the control of chaos in particular.

Keywords: Optimal monetary policy, Interest Rate Rules, Chaos Control, Endogenous fluctuations and Stabilization

*Corresponding author. Department of Economics, ISCTE, Lisbon, Portugal. Electronic address: *vivaldo.mendes@iscte.pt*.

†Department of Quantitative Methods, ISCTE, Lisbon. Address: ISCTE, Av. Forcas Armadas, 1649-026 Lisbon, Portugal. Electronic address: *diana.mendes@iscte.pt*.

"Grandmont [1985] argues that such deterministic cycles provide an alternative to the linear stochastic process view of cycles. If he is correct, *policy can have very drastic effects on the dynamics* of the economy by changing the specific form of nonlinearity."

Blanchard and Fisher (1989, 261).

The fact that the solutions are chaotic does not alone provide any justification for government intervention, *and indeed any such intervention could produce a stable, but Pareto inferior solution*. [Therefore] the existing theoretical results on chaos have no policy relevance, since in chaotic models the justification for intervention always can be identified *with a form of market failure* entered into the structure of the model, and hence the chaos is an independent and policy-irrelevant feature of those models. (page 44)

Barnett, Medio and Serlettis (1999)

1 Introduction

One of the most interesting facts in the fields of economics and finance over the last two decades consists of the finding that many simple dynamic general equilibrium models, using the most standard and innocuous assumptions, may lead to very complex dynamics, ranging from indeterminacy, large cyclical instability, and even bifurcation and route to chaos. These results are obtained in models where the economy has agents assumed to have rational and homogeneous expectations (or perfect foresight, depending on the case) and informationally efficient markets. Excellent surveys of this literature can be found in Brock (1997), Benhabib (1992), Boldrin and Woodford (1990), Guesnerie and Woodford (1992), and more recently those of Barnett *et al.* (1999) and Nishimura and Sorger (1999), among others.

However, the potentiality for very complex behavior becomes significantly increased if heterogeneous agents and different learning processes are also taken into account in the modelling of dynamic economic processes (e.g., Brock and Hommes, 1998; Chiarella and He, 2000; Evans and Honkapohja, 2001 and Saari 1996), or if we abandon the rather restrictive field of dynamic general equilibrium models. Nonoptimal dynamic models in the spirit of the early literature on business cycles, following the early contributions of Kaldor, Hicks or Goodwin, free from the standard representative optimal agent, are prone to produce irregular dynamics and chaos. See, eg., Puu (2000), Barkley Rosser (2004), Day (2000) and Goodwin (1992).

From the already impressive amount of literature on endogenous cycles and chaotic dynamics in economics, there are two major points that

should be stressed. Firstly, on the positive side of economic analysis, it became widely accepted that it is relatively easy to generate chaotic dynamics in macroeconomic models, even though this may come with a cost in some cases (relatively implausible parameter values) in highly aggregative models. At a time when rational expectations and real business cycles apparently dominated the field of macroeconomics, it came as a surprise that even the most strong versions of perfectly competitive markets could not produce the well behaved economic outcome that we learn in dominant textbooks.

Secondly, it is interesting to note that on the normative side we are not aware of an extensive literature dealing with the process of controlling chaotic economic dynamics (or some of control useful for economic policy), despite the understandable great importance of this type of control for analyzing the power of policy in nonlinear economic systems. In fact, the issue of economic policy was one of the fundamental points initially raised when chaotic dynamics started to be frequently and seriously discussed within the field of economics, as the sentence by Grandmont above shows very clearly.

The first strand of literature applying chaos control to economics includes papers which deal essentially with the study of dynamic oligopolistic games in a partial equilibrium framework. These include those of Holyst *et al.* (1996) and (2001), Kopel (1997) and more recently Matsumoto (2004). In the area of macroeconomics, Kaas (1998) applied control to a non-optimal conventional macroeconomic model and advised in the conclusion about the perils of applying control methods to optimal competitive frameworks.

None of these papers discusses the control of chaotic motion arising from a general equilibrium optimal dynamic process because it has become widely accepted in the economics profession that chaotic dynamics does not bring any new novelty to the analysis of economic policy to control business cycles. For example, Bullard and Butler (1993) and Barnett *et al.* (1999) — see one of the opening sentences above — argue forcefully that whenever agents optimize in a dynamic general equilibrium setting, and there are no market imperfections or incompleteness, if the result is chaotic dynamics with large and irregular cycles, the maximization of social welfare implies that the best policy is no activist policy at all. Therefore, following this argument, the power and beauty of chaotic dynamics brings no new secrets to the continual struggle of mankind to improve the way economic structures evolve over time.

More recently, a new strand of papers have tried to attack this issue. These include, on the side of fiscal policy, the papers by Guo and Lansing (2002) and (2004), Seegmuller (2003), Christiano and Harrison (1998),

Augeraud-Veron and Augier (2001), and Aloï, Jacobsen and Loyd-Braga (2001). Airaudo and Zanna (2004) deal with optimal monetary policy and Wieland and Westerhoff (2004) with chaos in exchange rate models. Despite the interesting aspects of all the papers above concerning the role of policy to control endogenous fluctuations, the only papers that in fact use the techniques of chaos control are those of Guo and Lansing, Aloï *et al.* and Wieland and Westerhoff. Airaudo and Zanna try to give an answer to the problem of controlling endogenous fluctuations in an optimal monetary policy framework, but in fact, they rule out chaotic dynamics by imposing a much larger perturbation than the one which is sufficient if we apply the techniques of chaos control.

In this paper we hope to clarify some issues related to the use of chaos control to reduce or eliminate business cycles. We do so by discussing chaotic dynamics in an optimal monetary model that has been studied in great detail by Benhabib *et al.* (2002). In particular, we make four basic points: (i) the model developed by Benhabib and associates suffers from serious drawbacks to be used as a theoretical benchmark to guide optimal monetary policy; (ii) the time span required to achieve successful control is generally small; (iii) ignorance of initial conditions are not a serious impediment to obtain control of the chaotic dynamics in the model; (iv) we argue that the conventional view of economic policy in nonlinear general equilibrium models — as the sentences above by Barnett *et al.* (1999) and Bullard and Buttler (1993) show very clearly — is based on a misconception of chaos in general, and on the control of chaos in particular.

The paper is organized as follows. In section 2, the dynamics of an optimal monetary policy model is studied in some detail, including stable steady states, periodic motion, bifurcations and chaos. Section 3 deals with the control of chaotic motion, and section 4 discusses the relevance of this type of control for economic policy. Section 5 concludes.

2 Optimal Monetary Policy and Endogenous Cycles

Since the early 1990s we have witnessed an increasing consensus in the conduct of modern monetary policy. Goodfriend and King (1997) have labelled this new consensus as "The New Neoclassical Synthesis and the Role of Monetary Policy", while Clarida, Gali and Gertler (1999) called it the "The Science of Monetary Policy: A New Keynesian Perspective". This new framework is a natural extension of the seminal idea developed by Taylor (1993), in which the central bank should conduct monetary policy through an aggressive and publicly known rule (Taylor Rules).

The central elements of this consensus are that:

- the crucial instrument of monetary policy ought to be the short term interest rate,
- policy should be focused on the control of inflation,
- inflation can be efficiently controlled by an aggressive increasing of short term interest rates,
- the central bank should conduct monetary policy adopting a strategy of commitment in a forward-looking environment (instead of discretion).

The fundamental objective of monetary policy in the framework above presented (or in any other one, we guess) is to reduce as much as possible the amplitude of business cycles, and by doing so increasing economic welfare. A huge amount of literature on Taylor rules seem to have confirmed the theoretical attractiveness of such rules, but also the empirical relevance of them. However, over the last two/three years, Benhabib, Schmitt-Grohé and Uribe have shown in a series of papers (2001a), (2001b), (2001c), (2002) and (2004) that active interest rules may lead to very unexpected consequences: indeterminacy, deflation traps, large cyclical instability, and even chaotic dynamics.

In their (2002) paper Benhabib and associates raise two fundamental points. Firstly, that under both active forward looking and active contemporaneous looking interest rate rules the calibration of the model lead to chaotic dynamics. Secondly, they show that the widely used local analysis of applying log linearization techniques around the steady state may produce wrong conclusions about the stability of the model under consideration. Finally, they leave the problem of policies to remedy the problem of endogenous cycles for further research.

2.1 The model

The model has the basic ingredients of a dynamic general equilibrium model in a discrete time framework. It is assumed for analytical convenience that besides households the only other agent in the model is the government (or a central bank).

There is a large number of households which maximize their lifetime utility given by the following utility function

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \quad (1)$$

in which, c_t stands for real consumption, and (σ, β) are parameters obeying the usual restrictions: $\sigma > 0$ and $\beta(0, 1)$.

The government prints money (M_t) and sells bonds to families (B_t). This leads to a budget constraint of the representative agent of the form

$$M_t + B_t + P_t c_t + P_t \tau_t = M_{t-1} + R_{t-1} B_{t-1} + P_t y_t \quad (2)$$

where P_t stands for the price level, τ_t for lump-sum taxes collected by the government, and y_t for the level of real income. Defining real balances by $m_t \equiv M_t/P_t$, financial wealth by $a_t \equiv (M_t + B_t)/P_t$, and the gross rate of inflation by $\pi_t \equiv P_t/P_{t-1}$, equation (2) can be written in real terms as

$$a_t + c_t + \tau_t = \frac{(1 - R_{t-1})}{\pi_t} m_{t-1} + \frac{R_{t-1}}{\pi_t} a_{t-1} + y_t \quad (3)$$

The stream of real income arises from a fixed productive factor ($\mathcal{B}$) and real balances (so money facilitates transactions) and it follows a CES type function

$$f(m_t) = [\mathcal{C} m_t^\mu + (1 - \mathcal{C}) \mathcal{B}^\mu]^{1/\mu} \quad , \quad \mu < 1, \mathcal{C} \in (0, 1) \quad (4)$$

Households maximize utility (1) subject to the two restrictions (2)-(4), from which there is an optimal plan of intertemporal sequences $\{c_t, m_t, y_t, a_t\}_{t=0}^\infty$. This leads to two conventional results given by the Euler equation and the static equality between the marginal productivity of money and its opportunity cost, given respectively, by

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \quad (5)$$

$$f'(m_t) = \frac{R_t - 1}{R_t} \quad (6)$$

Finally, in order to close the model, Benhabib *et al.* (2002) assume that the central bank conducts monetary policy with an optimal interest rate rule that obeys a feedback rule of the form

$$R_t \equiv \rho(\pi_{t+j}) = 1 + (R^* - 1) \left(\frac{\pi_{t+j}}{\pi^*} \right)^{\frac{A}{R^* - 1}} \quad , \quad j = 0, 1 \quad (7)$$

This function has several interesting points. Firstly, if we take $j = 1$ in equation (7) then $R_t = \rho(\pi_{t+1})$ and this means that the central bank sets forward-looking interest rate rules and if $j = 0$ then we have contemporaneous interest rate feedback rules.¹ Secondly, the feedback rule above satisfies a set of restrictions that have been established in the literature as commonly accepted:

¹In this paper, we will only discuss the case of forward-looking interest rates.

- The elasticity of the feedback rule at π^* is greater than unity (actually, equals to A/R^*), which means that the central bank reacts aggressively to inflationary pressures near the target rate of inflation;
- Liquidity traps (or the zero bound on nominal interest rates) are avoided due to the assumption that $\rho(\pi^*) > 1$;
- By assuming that $\rho(\pi^*) = \pi^*/\beta$, the existence of a fixed point consistent with the target rate of inflation is assured.

In order to obtain a closed form solution to the forward-looking interest rate rule version of the model, firstly, we need to combine equation (6) with (4), which leads to the following negative relation between output and the nominal interest rate:

$$R_t = R(y_t), \quad R' < 0. \quad (8)$$

Secondly, one has to use the Euler equation (5), the optimal interest rate equation (7), and the general equilibrium condition $y_t = c_t$, to finally arrive at a first order non-linear difference equation in output of the form:

$$y_{t+1} = F(y_t) = \beta^{\frac{1}{\sigma}} y_t \left(\frac{R(y_t)}{\rho^{-1}(R(y_t))} \right)^{\frac{1}{\sigma}}, \quad \sigma > 0, 0 < \beta < 1, \quad (9)$$

where $\rho^{-1}(\cdot)$ is the inverse of the function $\rho(\cdot)$. Finding an equilibrium real allocation reduces to finding a real positive sequence $\{y_t\}_{t \in \mathbb{N}}$ satisfying this last equation.

In order to proceed to the dynamical analysis of this difference equation or discrete map, we write down the explicit expression of F . So, since we have

$$R_t = R(y_t) = \frac{1}{1 - \mathcal{C} \left(\frac{y_t^\mu - (1 - \mathcal{C}) \mathcal{B}^\mu}{\mathcal{C} y_t^\mu} \right)^{\frac{\mu-1}{\mu}}} \quad (10)$$

and as $\pi_{t+1} = \rho^{-1}(R(y_t)) = \pi^* \left(\frac{R_t - 1}{R^* - 1} \right)^{\frac{R^* - 1}{A}}$, then

$$\pi_{t+1} = \pi^* \left(\frac{\frac{1}{1 - \mathcal{C} \left(\frac{y_t^\mu - (1 - \mathcal{C}) \mathcal{B}^\mu}{\mathcal{C} y_t^\mu} \right)^{\frac{\mu-1}{\mu}}} - 1}{R^* - 1} \right)^{\frac{R^* - 1}{A}}$$

By substituting these two expressions in (9) we obtain the following non-linear one-dimensional map from which we can find the real allocation equilibrium:

$$y_{t+1} = F(y_t) \equiv \beta^{1/\sigma} y_t \left(\frac{\frac{1}{1 - C \left(\frac{y_t^\mu - (1 - C) \mathcal{B}^\mu}{C y_t^\mu} \right)^{\frac{\mu-1}{\mu}}}}{\pi^* \left(\frac{\frac{1}{1 - C \left(\frac{y_t^\mu - (1 - C) \mathcal{B}^\mu}{C y_t^\mu} \right)^{\frac{\mu-1}{\mu}}} - 1}{R^* - 1} \right)^{\frac{R^*-1}{A}}} \right)^{\frac{1}{\sigma}}, \quad (11)$$

with $\sigma > 0$, $0 < \beta < 1$, $\mu < 1$, $0 < C \leq 1$, $A/R^* > 1$, $R^* > 1$, $\pi^* > \beta$.

The graphical representation of the map F is illustrate in *Figure 1*. There are two equilibrium points, denoted by y_A^* and y_P^* . The active steady state y_A^* has a more interesting dynamical behavior, and a detailed nonlinear analysis shows that this equilibrium point has a very complex dynamics, from a stable equilibrium to chaotic orbit, when one or various parameters are varied (in this paper we study only variations in one of the parameters: A). We are interest in a small neighborhood around the steady state, where the perfect-foresight equilibrium real allocations remain stable. In order to obtain this we show that, for certain parameter calibration, the map F is a typical unimodal map and this means that the analysis of the unique critical point y_c will characterize the global behavior of the system. Hence, we obtain that y_A^* is globally stable for certain parameter calibration. Moreover, when the active fixed point becomes erratic, we proceed in the next section to control the chaotic motion in order to achieve stability for a larger set of system parameters.

2.2 Chaotic Dynamics

We have the following parameter calibration: $\beta = 0.996$, $\sigma = 1.5$, $\mu = -9$, $C = 0.000352$, $\mathcal{B} = 1$, $\pi^* = 1.0103$, $R^* = 1.0147$, and let the parameter A

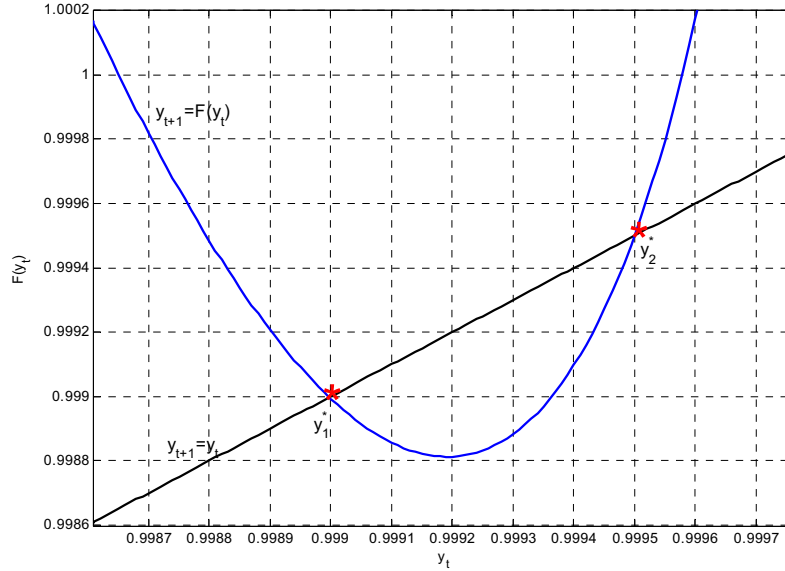


Figure 1: The map $F(y_t)$, the 45° diagonal and the two equilibrium points

to vary in the interval $(1.3, 1.55)$.² The bifurcation diagram of the map F when A is varied in the above interval it is presented in *Figure 2*. The first bifurcation point appears for $A = 1.408\dots$ and after this value a period-doubling route to chaos takes place. The bifurcation diagram is typical of a unimodal or a quadratic map and, hence, the period three orbit implies chaos, following the Li–Yorke theorem. Actually, for any parameter value after the bifurcation point of the 2^∞ orbit, we have chaotic motion, which can also be applied to the value of A after such bifurcation point.

Proposition 1 *For the above parameter calibration and for $1.3 < A < 1.408$ the active steady state y_A^* is globally stable.*

Proof. To prove this it is sufficient to recall that the map F is unimodal in the invariant interval I , $I = [F(y_c), F^2(y_c)]$.³ The critical point y_c , that is, the solution of the equation $F'(y_t) = 0$, always converge to the unique stable steady state of the map, in this case y_A^* . This is also illustrate in *Figure 3*.

²In the original paper [10], the parameter A is fixed at the 1.522 and the parameter σ is varied.

³Basically, a map is called unimodal if it has a unique critical point C in the interval $[A, B]$ and is decreasing (increasing) in the interval $[A, C]$ and increasing (decreasing) in the interval $[C, B]$.

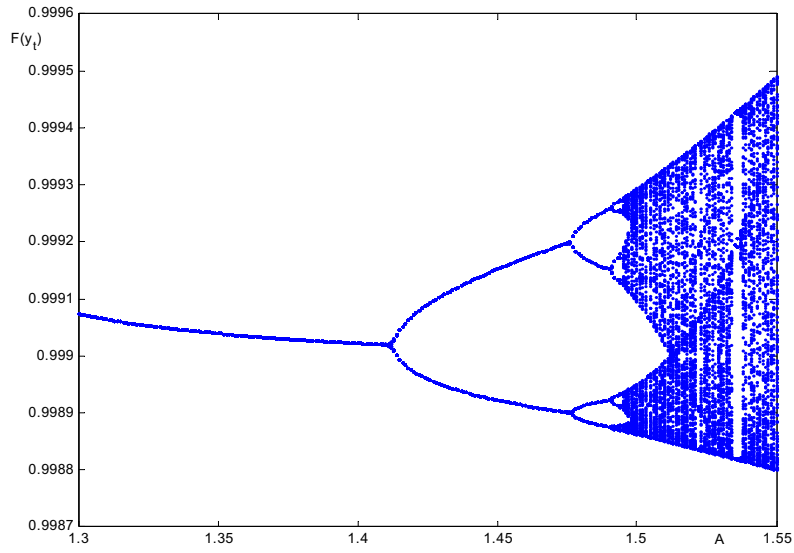


Figure 2: *Bifurcation diagram for parameter A .*

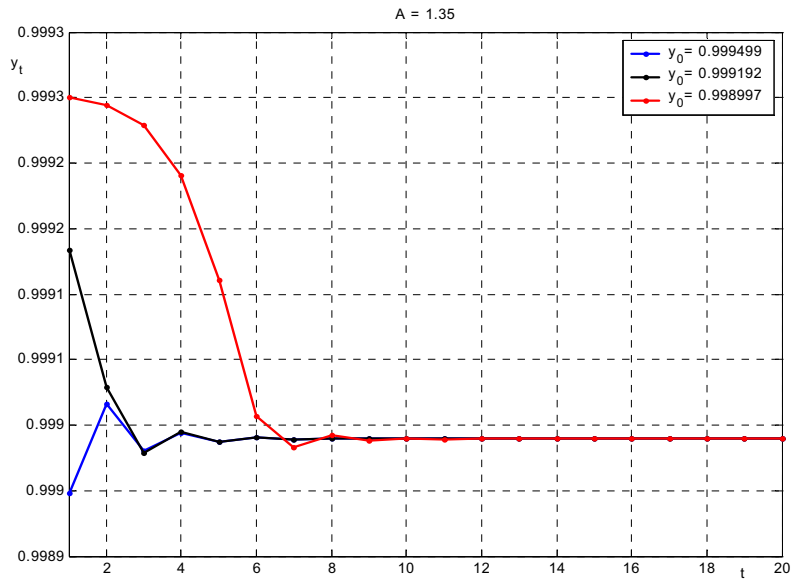


Figure 3: *Stability of the steady state y_A^* .*

The stability interval is lower bounded by 1.3, the value for which the map F assumes real values, and is upper bounded by 1.408, that is, the value where the first period doubling bifurcation appears, $(\partial F/\partial y_t) = -1$. For any value within this interval, the passive fixed point is unstable. ■

Now, we fix $A = 1.54$, and we can observe from the bifurcation diagram (*Figure 2*) that for this parameter value the active steady state is chaotic. Namely, the equation $F(y_t) = y_t$ gives the two equilibria of these difference equation, that is, $y_A^* = 0.9989977$ and $y_P^* = 0.99949946$ which are shown in *Figure 1*. Both of them are unstable, since

$$\left| \left(\frac{\partial F}{\partial y} \right)_{(y_A^*)} \right| = 1.78858 > 1 \text{ and } \left| \left(\frac{\partial F}{\partial y} \right)_{(y_P^*)} \right| = 5.21621 > 1.$$

The unstable steady state y_A^* will be stabilized in the next section, by using OGY control technique.

3 Controlling Endogenous Cycles

We intend to show that chaotic dynamics may have a significant importance for economics not only on the modelling side, but also on the normative side by giving a possible new dimension to economic policy. To clarify this point we will apply the OGY method [40] to control the chaotic motion that is produced in this model.

Let us note the map by $F(y_t, A_t)$ in order to control the unstable period one orbit by applying a tiny perturbation to A , which is assumed as the parameter controlled by the government or the central bank.⁴ The control strategy is the following: find a stabilizing local feedback control law which is defined on a neighborhood of the desired periodic orbit. This is done by considering the first order approximation of the system at the chosen unstable periodic orbit. The ergodic nature of the chaotic dynamics of the model ensures that the state trajectory eventually enters into the neighborhood. Once inside the neighborhood, we apply the stabilizing feedback control law in order to steer the trajectory towards the desired orbit.

For values of y_t close to the unstable fixed point y_* and for values of A_t close to A_* , the map F can be approximated by the following linear discrete time system

$$x_{t+1} = Bx_t + Cz_t, \tag{12}$$

where $x_t = y_t - y_*$ and $z_t = A_t - A_*$ are the derivations from the nominal values in standard control notation for states and input. The values B

⁴We develop further this issue (why this parameter is the instrument of monetary policy) in the next section.

and C represent the derivatives of the map F with respect to the variable y and to the control parameter A evaluated at the point (y_*, A_*) , that is

$$B = \left(\frac{\partial F}{\partial y} \right)_{(y_*, A_*)} \quad \text{and} \quad C = \left(\frac{\partial F}{\partial A} \right)_{(y_*, A_*)}.$$

Now according to OGY a linear state feedback

$$z_t = -Kx_t$$

can be applied to system (12). It should be added that this control should only be applied within a certain region

$$\mathcal{R}_\varepsilon = \{y : |y - y_*| < \varepsilon\}, \varepsilon > 0$$

around the fixed point, which it is called the control region. Then, the system (12) will take the form

$$x_{t+1} = (B - CK)x_t,$$

and thus the closed loop system is stable as long as

$$|(B - CK)| < 1.$$

Setting $(B - CK) = 0$, then we have the pole placement technique and obviously $K = B/C$.

It was shown in the previous section that for $\beta = 0.996$, $\sigma = 1.5$, $\mu = -9$, $\mathcal{C} = 0.000352$, $\pi^* = 1.0103$, $R^* = 1.0147$ and $A = 1.54$ the map F possesses an unstable chaotic fixed point $y_A^* = y_* = 0.9989977\dots$. We fix these parameter values and consider that A is the control parameter which is available for external adjustment but restricted to lie in some small interval $|A - A_*| < \varepsilon, \varepsilon > 0$ around the nominal value $A_* = 1.54$. Since

$$B = \left(\frac{\partial F}{\partial y} \right)_{(y_*, A_*)} = -1.78858\dots \quad \text{and} \quad C = \left(\frac{\partial F}{\partial A} \right)_{(y_*, A_*)} = -0.0003162\dots,$$

one obtains that

$$x_{t+1} = (-1.78858 + 0.0003162K)x_t,$$

which means that the parameter K may take values in the interval $(2493.8098, 8818.5767)$ which is the solution to the stability inequality

$$|-1.78858 + 0.0003162K| < 1.$$

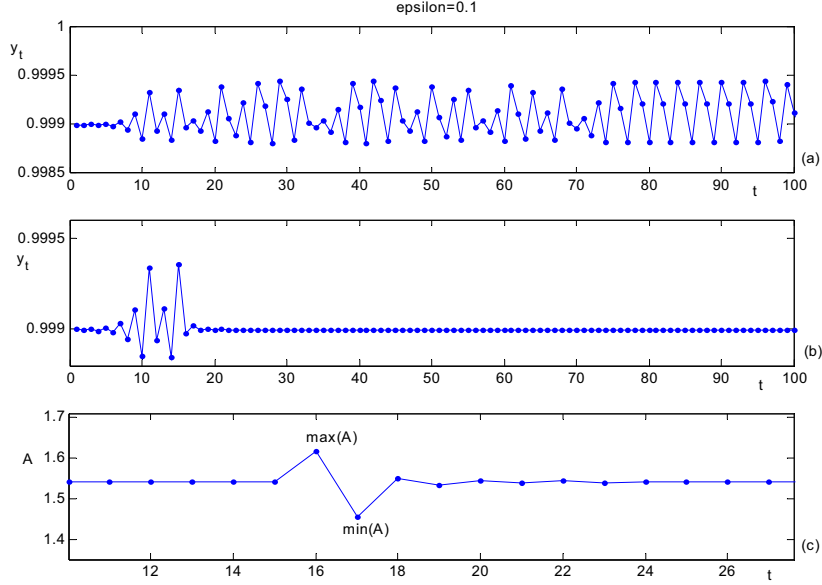


Figure 4: (a) The chaotic unstable orbit to be controlled; (b) the controlled orbit; and (c) the variation of the control parameter A

Obviously, if we choose the pole placement value, that is $-1.78858 + 0.0003162K = 0$, it follows that $K = 5656.483238$. For this value of the control constant K , the unstable period one orbit is stabilized, as one can easily see in Figure 4, panel (b). Panel (a) shows the randomly chosen trajectory which we wish to steer towards the fixed point. In this case we choose $\varepsilon = 0.1$, and the variation of the parameter A is shown in panel (c). The time span required to achieve control is very short, independently of the initial condition, the moment when the control is switched on (generally after the first 10 iterations) or the value of K . Figure 5 illustrates control for several choices of the parameter K and two different initial conditions y_0 and y'_0 .

In the cases illustrated in Figures 6 and 7 we choose $\varepsilon = 0.02$ and $\varepsilon = 0.001$. The variations of the control parameters are very tiny $|A - A^*| < \varepsilon$, namely in these concrete cases $1.5327 < A^* < 1.5446$ and $1.5397 < A^* < 1.5404$. From a purely mathematical point of view, this model is quite easy to control and this is obvious from the two previous figures, where a relative short time span is required in order to achieve stabilization, even when the permitted variation interval for the control parameter is very small.

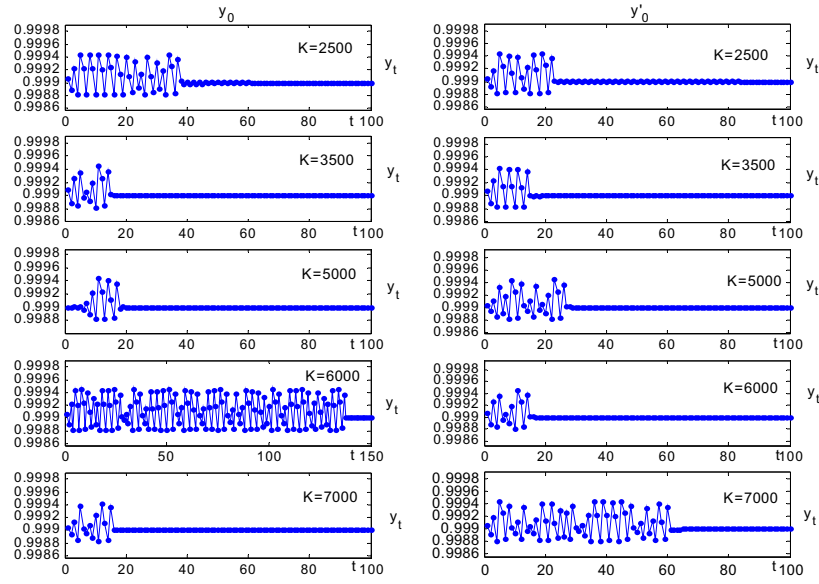


Figure 5: The variation of control for two different initial values and 5 different values for K

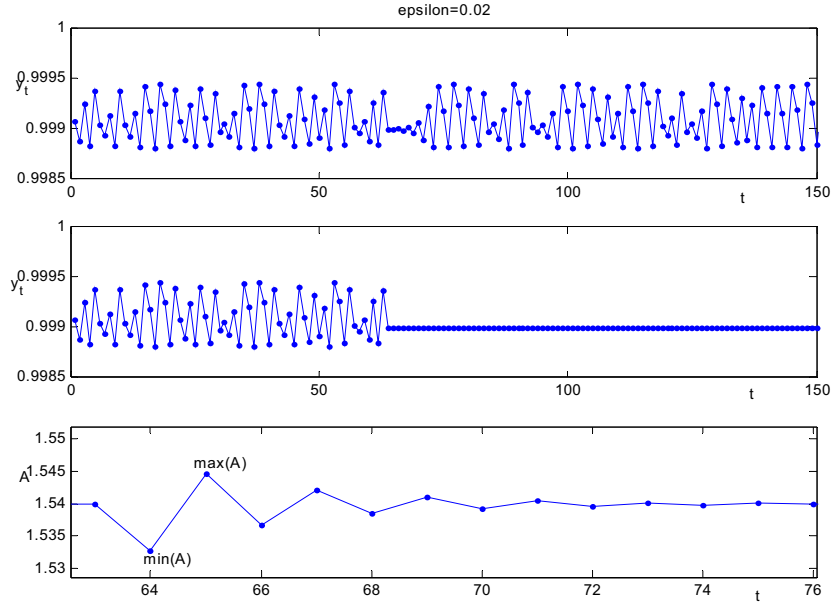


Figure 6: Control with $\varepsilon = 0.02$

4 Relevance for Economic Policy

The fundamental importance of optimal monetary policy, conducted in accordance with active interest rules, is to reduce the amplitude of business cycles and, by doing so, increasing economic welfare. It seems therefore ironic that a policy which is designed in principle to reduce or eliminate business cycles is capable, by itself, to produce large and endogenous fluctuations.

Benhabib *et al.* (2002) did not enter into the issue of policy measures to remedy this drawback of active interest rate rules. In fact, in an earlier version of the paper, they state that "the design of policies capable of eliminating chaotic dynamics remains a subject for future research"⁵ In what follows we will try to clarify some of the points related to this issue, and we will concentrate on two fundamental points: (i) the relevance of the present model as a possible guide to effective monetary policy; and (ii) the relevance of chaotic dynamics (and chaos control) to economic policy.

As far as the first point is concerned, it is unfortunate that the model

⁵From page 17 of a larger version of the paper, initially presented in 2001 at a NBER meeting.

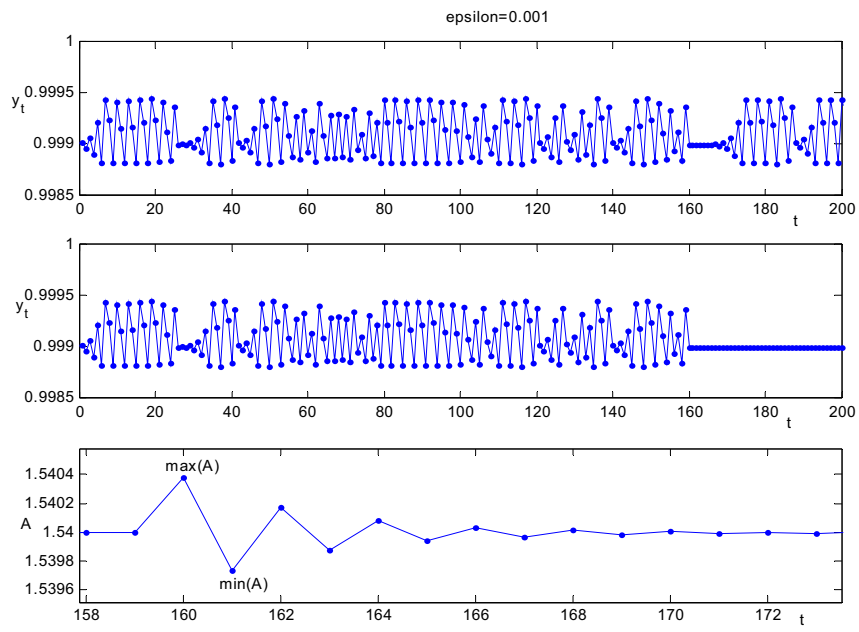


Figure 7: Control with $\varepsilon = 0.001$

put forward by Benhabib and associates is of little help to be a theoretical reference for guiding optimal monetary policy. The reason is very simple: one of the major results of the model, however totally ignored by Benhabib *et al.*, is totally at odds with the spirit of active interest rate rules (Taylor rules) and basic economic intuition. This result can be easily observed by a simple inspection of the bifurcation diagram in *Figure 2*. This figure shows the evolution of the stability of the model with respect to the fundamental policy parameter (A). This parameter gives the aggressiveness of the central bank towards inflation pressures near the target inflation rate (π^*), such that an increase in the value of A represents less tolerance of the central bank towards inflation. Therefore, an increase in A should lead to an increase in the stability of the dynamics of the model economy.

Unfortunately, this does not happen. In fact, it came out as a great surprise, that once the bifurcation diagram is produced one can easily find out that an increase in the policy parameter A leads to an increase in instability following the already traditional period doubling bifurcation. For low values of A , one may get a stable fixed point, and for higher values it degenerates into period-two cycles, period-four, and so on, until chaotic motion sets in. This occurs at the following value $A > 1.408$. Therefore, the model, as it stands at the moment, is of little help for optimal monetary policy.

On the second point above mentioned — as far as the practical relevance of chaotic methods for economic policy is concerned — there have been three questions raised about this issue:

- How many periods of time does the control require in order to take place?

We have received some feedback arguing that, as the control of chaotic dynamics usually takes a long period of time to take effect, then, the former is of little practical relevance for economic policy in the world we live in today. This argument may be valid for specific chaotic motions, however, as far as most economic models are concerned it seems misleading. In fact, in most control exercises we have undertaken using quarters as a time index, as we do in the model discussed in this paper, the control can take no more than just 5, 6 or 10 quarters, if the perturbation is not too small. In *Figure 8*, we recall two control exercises. In the first panel, it takes just 6 periods (quarters) if we accept a relatively not too large perturbation ($\varepsilon = 0.1$), while in the second panel it takes around 10 periods to have a successful control because the perturbation is much smaller ($\varepsilon = 0.02$). So the first conclusion concerning the relevance for practical economic policy is this: if one produces a large perturbation to

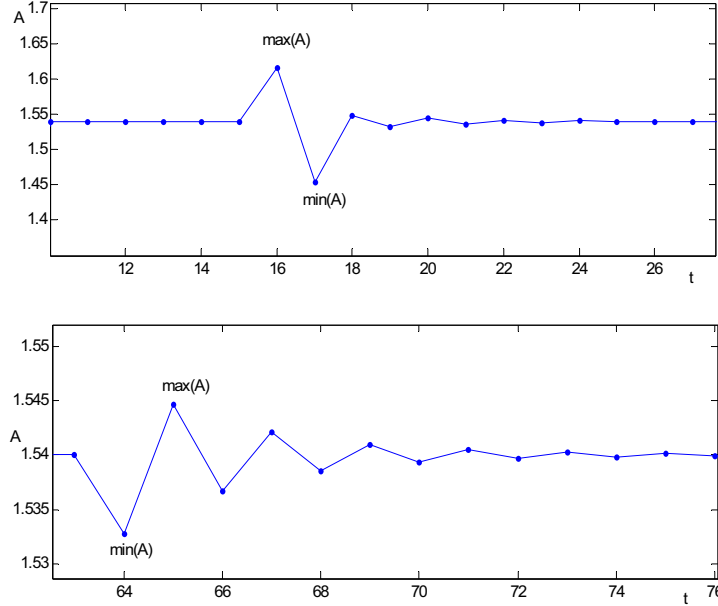


Figure 8: *Time span required to achieve successful control: panel (a) with a perturbation of $\varepsilon = 0.1$; panel (b) with a smaller perturbation $\varepsilon = 0.02$.*

the system dynamics, then we have a relatively short period of time to perform the control, if only a tiny perturbation is allowed, the time span is certainly larger.

- How can one control a chaotic system if this is so sensitive so small disturbances (mistakes)?

This is a point that is partially misdirected. It is precisely because the system is so sensitive to small perturbations (mistakes), that one can control it with techniques that can not be successfully applied to non-linear nonchaotic systems. Contrary to what happens with these latter processes, in chaotic dynamics we know that the orbits come back time and time again to the neighborhood of the unstable fixed point. A small perturbation can so be applied to force the system to remain at that fixed point. With linear stochastic processes, if the fixed point is stable, fluctuations are entirely due to exogenous and uncorrelated shocks. In this case policy is totally useless, because nothing can be done to prevent those shocks to occur. In this case, if a policy action is taken, then the

fixed point is changed (not its stability, but rather its coordinates), which does not happen in the case of controlling chaotic dynamics.

Nevertheless, there is a cost that has to be paid to cover for our ignorance of the true initial conditions of a chaotic process. In the control exercises we present in the paper we show that the fact that the ignorance of the particular initial condition of the system does not prevent us to obtain a successful control; however, the time span required to do so may be very different if the initial conditions are also very different. In *Figure 5* we choose two random initial conditions (y_0, y_0') , the system could be controlled, but for the same perturbations successful control requires different time spans.

Contrary to the two objections above, there is one side of chaos control applied to economics that may raise serious questions on its empirical relevance. As we saw in *Figure 8*, in order to control the endogenous fluctuations, the central bank has force both an increase and a decrease (or vice versa) in the short term interest rate in successive periods. Given what we know from the behavior of central banks over the last two decades, it seems to us that this behavior of the control rule does not comply in any way with the evidence of recent monetary policy in advanced economies. Nevertheless, we are not sure that techniques of chaos control (others than the OGY technique) are not able to overcome this shortcoming. This is left for future investigation.

- In dynamic general equilibrium models, when endogenous fluctuations exist, the associated policy advice is *laissez-faire*.

In a paper which particular focus is the link between chaos and economic policy, Bullard and Butler (1993) put the argument forward in a very clear fashion:

"It remains that there is no published example of a well specified, optimizing model, obeying baseline assumptions, where Pareto-inferior endogenous fluctuations exist. The reason for this seems clear – one must allow for some type of market incompleteness to justify government intervention under a criterion of Pareto optimality ... [therefore] unless one is willing to accept variations on the baseline assumptions ... the preliminary conclusion seems to be that when endogenous fluctuations exist in optimizing models, the associated policy advice is *laissez-faire*" (page 859).

An identical point is also put forward by Barnett *et al.* (1999). We argue that this view of economic policy in nonlinear general equilibrium models is based on a misconception of chaos in general, and on the control of chaos in particular.

The fact that chaotic systems have the fundamental characteristic of topological transitivity, which is not shared by nonlinear non-chaotic systems, one can control a chaotic system without changing its fundamental topological characteristics, as we showed above. The implications from topological transitivity are simple: by applying tiny perturbations to the system, we can stabilize an unstable fixed point, turning it into a stable fixed point; or we could also change an unstable orbit of period two into a stable periodic orbit of period two (but not of period three or of period one). This is in clear contrast to what happens with the control of nonlinear non-chaotic models, where the control in fact alters not only its stability but also the coordinates of fixed point (if there is one).

Therefore, if the fixed point remains the same, only its locally stability is altered, it seems questionable to argue, as Barnett *et al.* and Bullard and Butler have done, that cycles around an unstable fixed point are Pareto superior to a stable trajectory determined by the exactly same fixed point. But we understand the direct comparison that those authors establish between the case of a linear stochastic process and nonlinear chaotic one. In fact, the force of such argument implies that the irregular motion around the linear trend would lead to a higher Pareto ranking than any effort by the public authorities to reduce the amplitude of the exogenous cycles, because these cycles are not caused by any internal force of the economic process. Therefore, in linear stochastic processes, public authorities should intervene whenever there is some form of market failure which prevent private agents to quickly react to exogenous shocks by changing prices immediately (of labour, goods and services or interest rates).

However, this is not the case in chaotic economic dynamics. In this case, it is the very behaviour of economic agents that causes the economy to follow erratic cycles and, therefore, if these cycles can be somehow reduced or prevented economic agents do not have to react to the ups and downs of economic activity and their economic welfare is not altered: not reduced, nor increased.

5 Concluding Remarks

In this paper we discussed optimal monetary policy in a model which produces chaotic dynamics. Since the early 1990s we have witnessed an increasing consensus in the conduct of modern monetary policy. This new framework is a natural extension of the seminal idea developed by Taylor (1993), in which the central bank should conduct monetary policy through an aggressive and publicly known rule (Taylor Rules). The fundamental objective of monetary policy in such framework is to reduce

as much as possible the amplitude of business cycles, and by doing so increasing economic welfare.

However, over the last two/three years, Benhabib, Schmitt-Grohé and Uribe have shown in a series of papers that active interest rules may lead to very unexpected consequences: indeterminacy, deflation traps, large cyclical instability, and even chaotic dynamics. Therefore we have a puzzle: the basic objective of active interest rules is to stabilize the economy, but instead they lead to irregular cycles even in the absence of any form of exogenous shocks.

We wish to clarify some issues related to the use of chaos control to reduce or eliminate business cycles. In particular, we make four basic points: (i) the model developed by Benhabib and associates suffers from serious drawbacks to be used as a theoretical benchmark to guide optimal monetary policy, as the more aggressive the central bank becomes, the more unstable the economy will be; (ii) the time span required to achieve successful control is generally small; (iii) ignorance about the true state of initial conditions are not a serious impediment to obtain control of the chaotic dynamics in the model; (iv) we argue that the conventional view of economic policy in nonlinear general equilibrium models — when endogenous fluctuations exist in optimizing models, the associated policy advice is *laissez-faire* — is based on a misconception of chaos in general, and on the control of chaos in particular.

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